TOPOLOGICAL INSULATORS

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Emergence of Quantum Phases in Novel Materials
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Instituto de Ciencia de Materiales de Madrid

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AGENCIA ESTATAL DE INVESTIGACIÓN

GOBIERNO DE ESPAÑA MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

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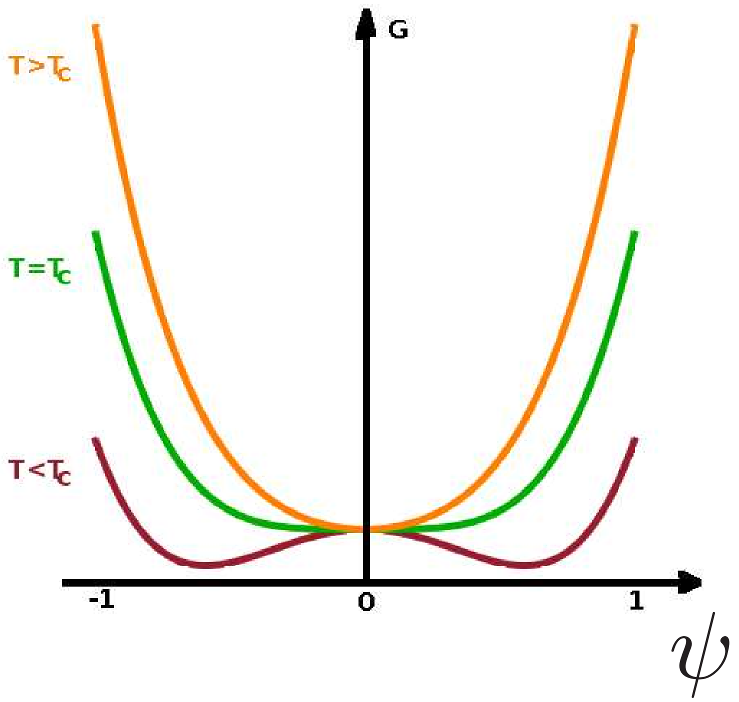
OUTLINE

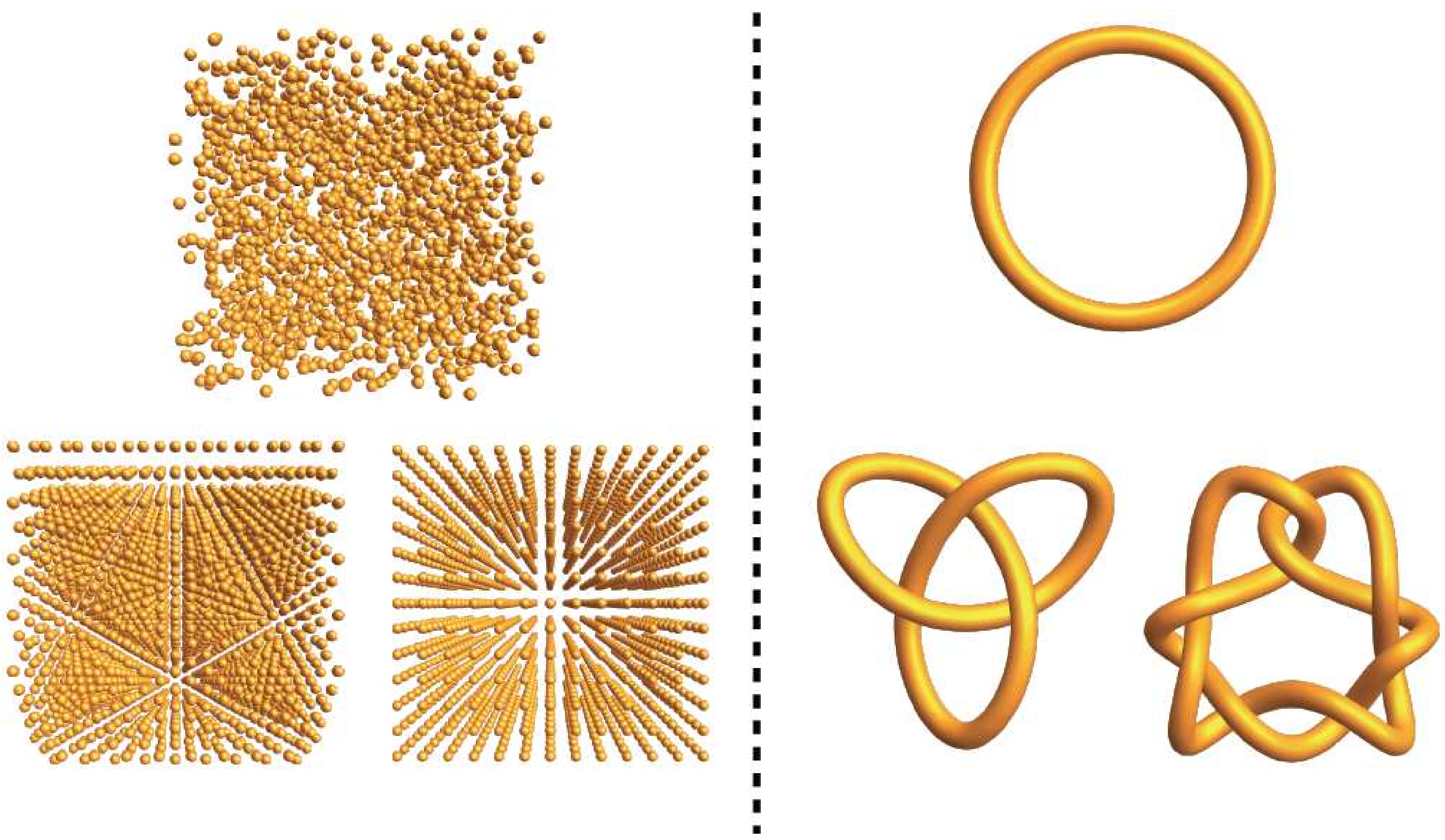
1. Topological order as a new paradigm.
2. Topology in one shot. Concept of SPTs.
3. Berry phase: "The mother of the lamb"
4. Time reversal invariant topological insulators
5. Conclusions to part II

1. Topological order as a new paradigm: local vs global

Order parameter ψ

$$F := \int d^D x \;\left(a(T) + r(T)\psi^2(x) + s(T)\psi^4(x) \; + f(T)(\nabla\psi(x))^2 \; + h(x)\psi(x) \; + \mathcal{O}(\psi^6; (\nabla\psi)^4)\right)$$





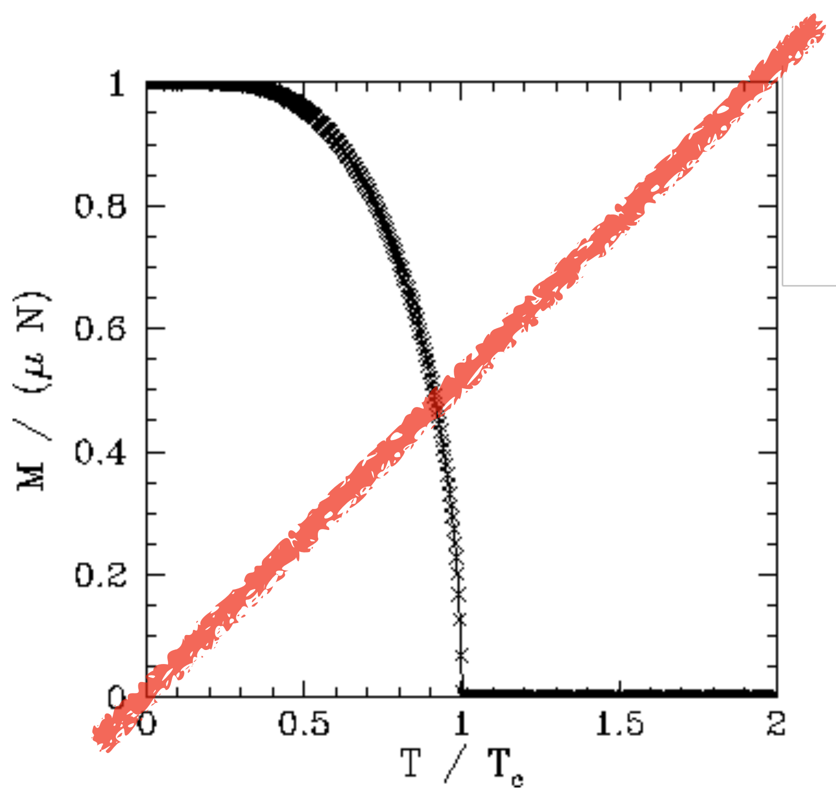
Global order parameter?
Global properties?

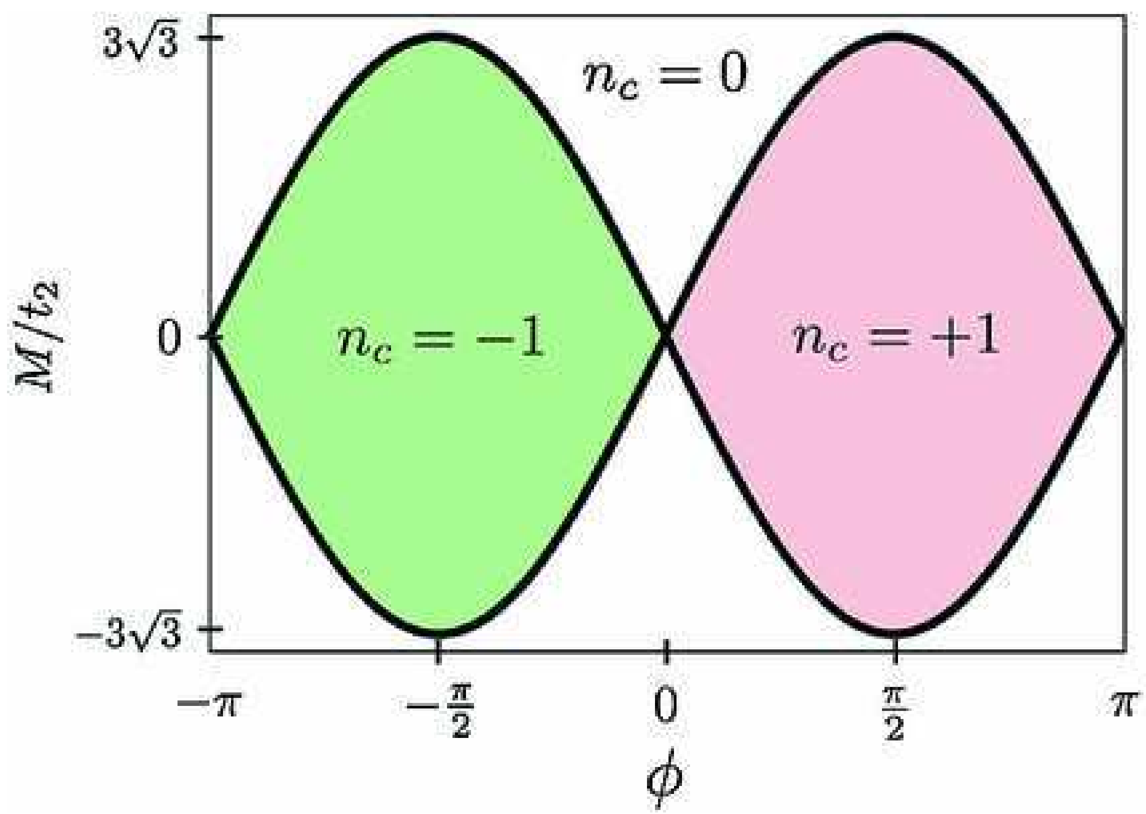
1. Topological order as a new paradigm: local vs global

In the QHE or the 2D Dirac equation (QAHE)

$$\sigma_H = \frac{e^2}{h} N$$

N is an integer, it does not change when adding small fluctuations





Haldane model phase diagram

1. Topological order as a new paradigm: local vs global

We had local fields, free energies, and so to describe different phases

Now we need quantities that do not change upon local fluctuations

Topology offers these quantities

Gauss-Bonnet theorem: Euler characteristic

$$\int_{\mathcal{M}} K dA = 2\pi\chi(\mathcal{M}) = 4\pi(1-g)$$

Number of holes

(Gaussian) Curvature

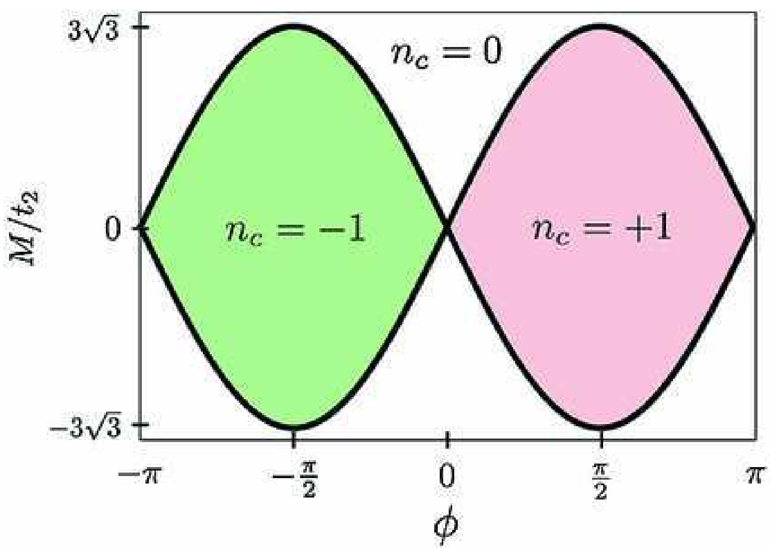
II. Definition of symmetry protected topological phases

SPTs: Zero temperature quantum states of matter having a symmetry and an energy gap that

i) Different SPTs with a given symmetry cannot be deformed into each other preserving The symmetry without closing the gap.
ii) Different SPTs can be deformed into each other without closing the gap if the Deformation breaks the symmetry.

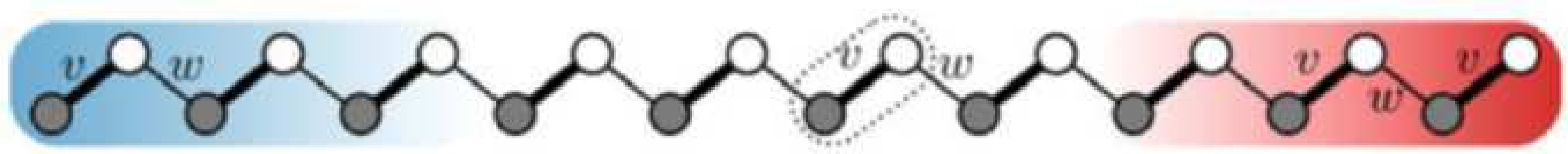
Each SPT is characterized by the value of a topological invariant

$$\sigma_H = \frac{e^2}{h} N$$



II. Definition of symmetry protected topological phases

Example: 1D Su-Schrieffer-Heeger model



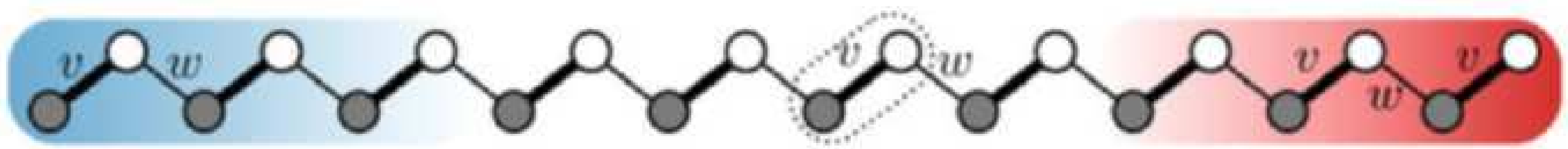
$$H = \sum_n v c^{+}_{A,n} c_{B,n} + w c^{+}_{A,n+1} c_{b,n} + h.c.$$

$$\psi_k = (c_{A,k}, c_{B,k})$$

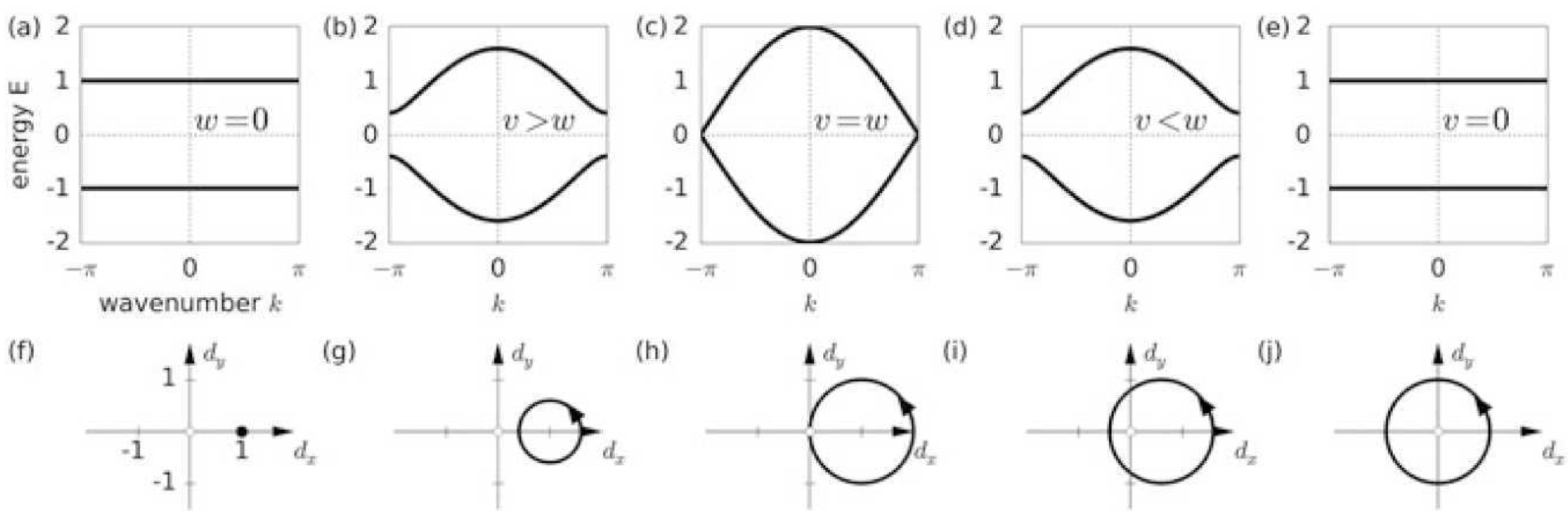
$$H = \sum_k \psi_k^{+} [(v + w\cos k)\sigma_1 + w \sin k \sigma_2] \psi_k = \sum_k \psi_k^{+} (\boldsymbol{\sigma} \cdot \mathbf{d}(k)) \psi_k$$

II. Definition of symmetry protected topological phases

Example: 1D Su-Schrieffer-Heeger model



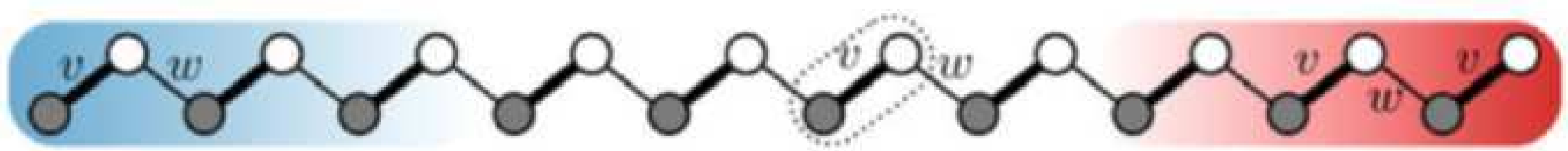
$$H = \sum_k \psi_k^+ [(v + w\cos k)\sigma_1 + w \sin k \sigma_2]\psi_k = \sum_k \psi_k^+ (\boldsymbol{\sigma} \cdot \boldsymbol{d}(k))\psi_k$$



Insulator (gapped) (gapless) Insulator (gapped)

II. Definition of symmetry protected topological phases

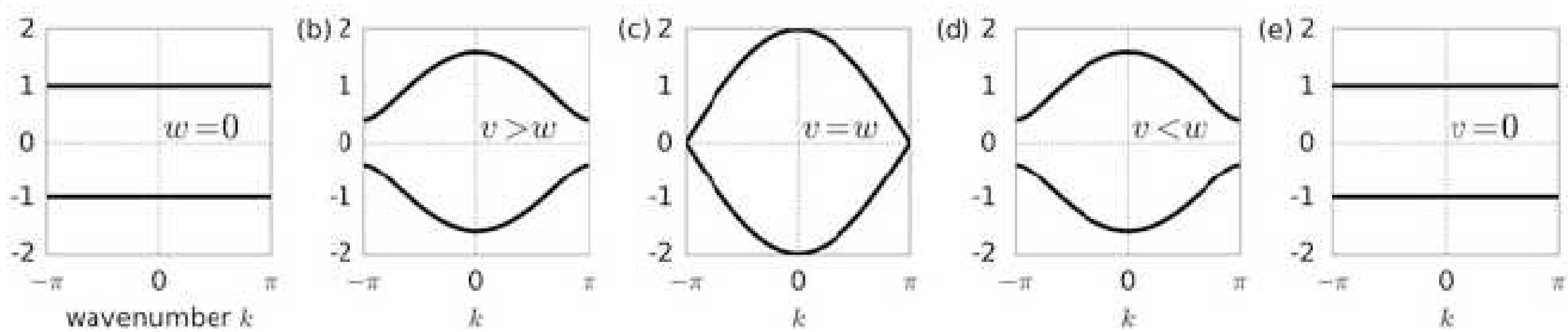
If a SPT, which symmetry goes with the system?



In this model we have not a symmetry but a chiral “symmetry”

$$[H,U] \neq 0 \qquad HU = -UH \qquad U = \sigma_3$$

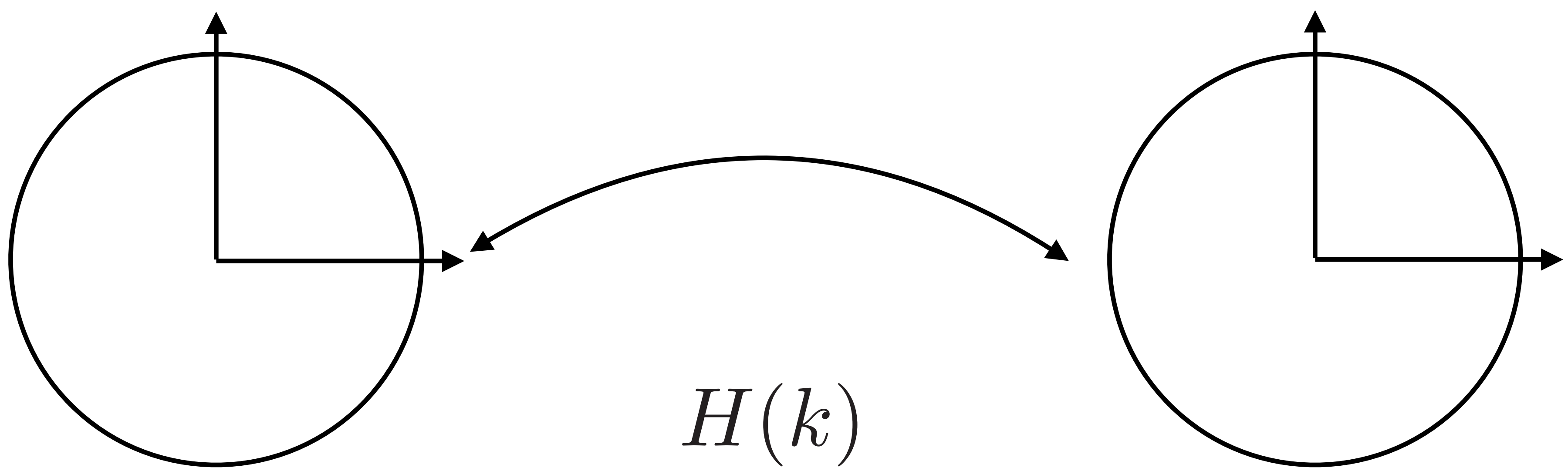
As long as the chiral symmetry is fulfilled, $d_3 = 0$



II. Definition of symmetry protected topological phases

How we define a topological invariant in the SSH model?

$$H(k) = \boldsymbol{\sigma} \cdot \boldsymbol{d}_k \qquad d_3 = 0$$



$$H(k)$$

$$k \in [0, 2\pi) \qquad \boldsymbol{n}_k = \frac{\boldsymbol{d}_k}{|\boldsymbol{d}_k|}$$

II. Definition of symmetry protected topological phases

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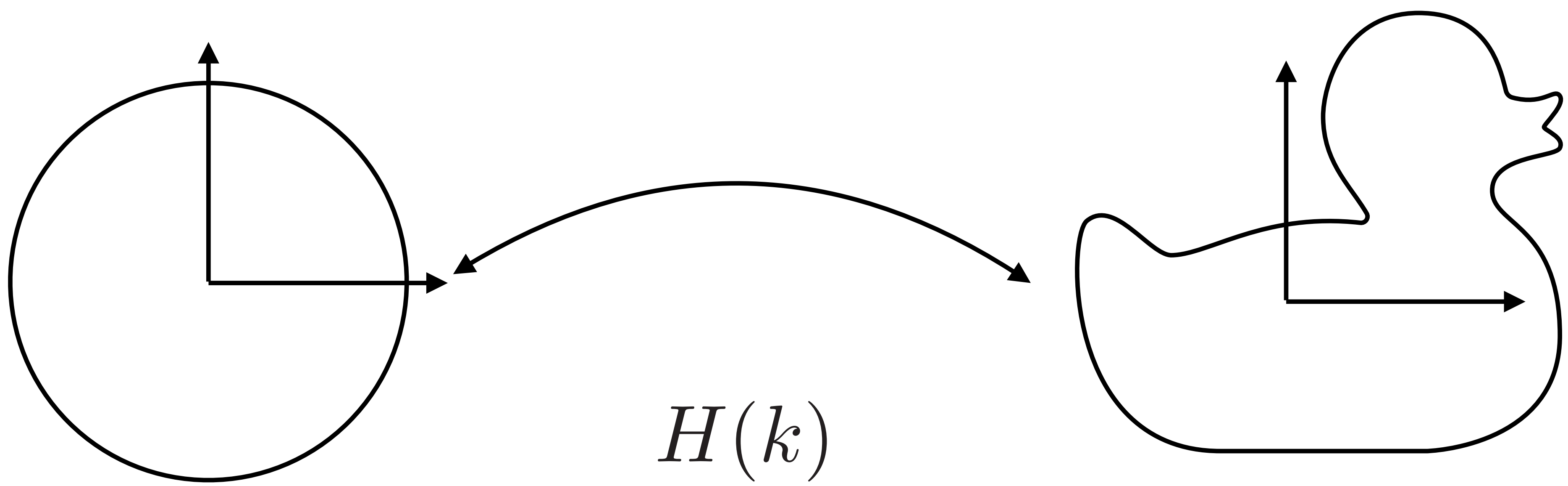

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II. Definition of symmetry protected topological phases

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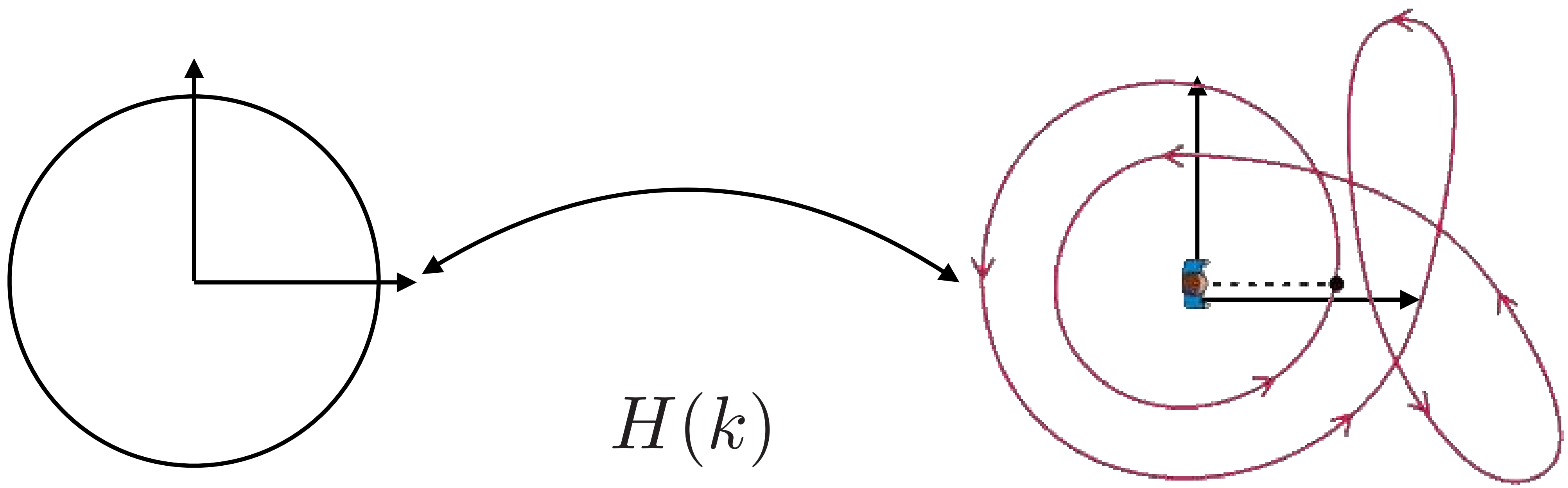


$$H(k)$$

$$k \in [0, 2\pi) \qquad \boldsymbol{n}_k = \frac{\boldsymbol{d}_k}{|\boldsymbol{d}_k|}$$

The application H defines a winding number as long as $\boldsymbol{n}_k$ is in the plane

II. Definition of symmetry protected topological phases

How we define a topological invariant in the SSH model?

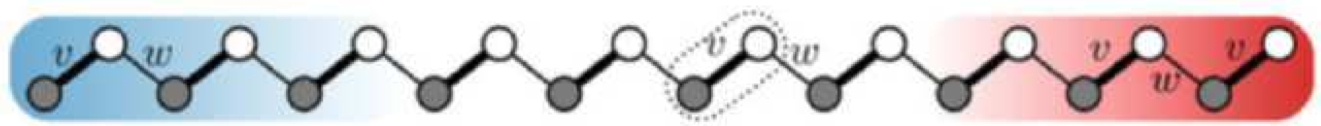
$$H(k) = \boldsymbol{\sigma} \cdot \boldsymbol{d}_k \qquad d_3 = 0 \qquad \boldsymbol{n}_k = \frac{\boldsymbol{d}_k}{|\boldsymbol{d}_k|}$$

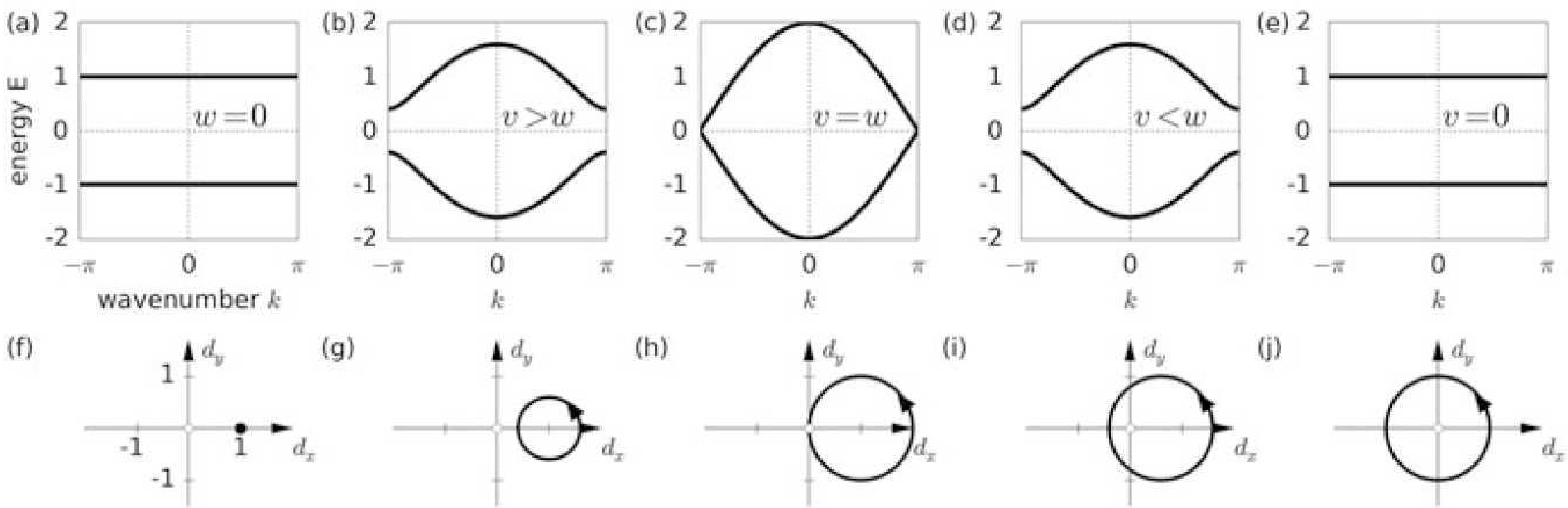
$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\boldsymbol{n}_k \times \partial_k \boldsymbol{n}_k) \cdot \boldsymbol{z} dk$$

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} i \langle s | \partial_k | s \rangle \, dk \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{A}_k dk$$

Berry connection

II. Definition of symmetry protected topological phases





Insulator (gapped)

(gapless)

Insulator (gapped)

$\nu = 0$

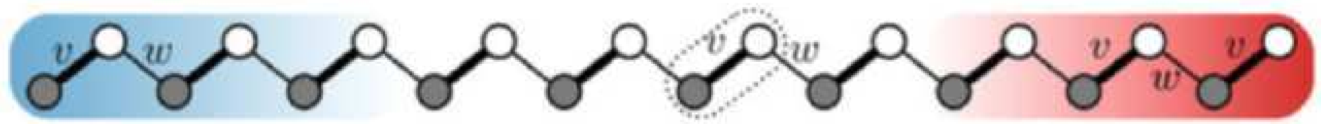
$\nu = ?$

$\nu = 1$

Trivial

Non trivial

II. Definition of symmetry protected topological phases

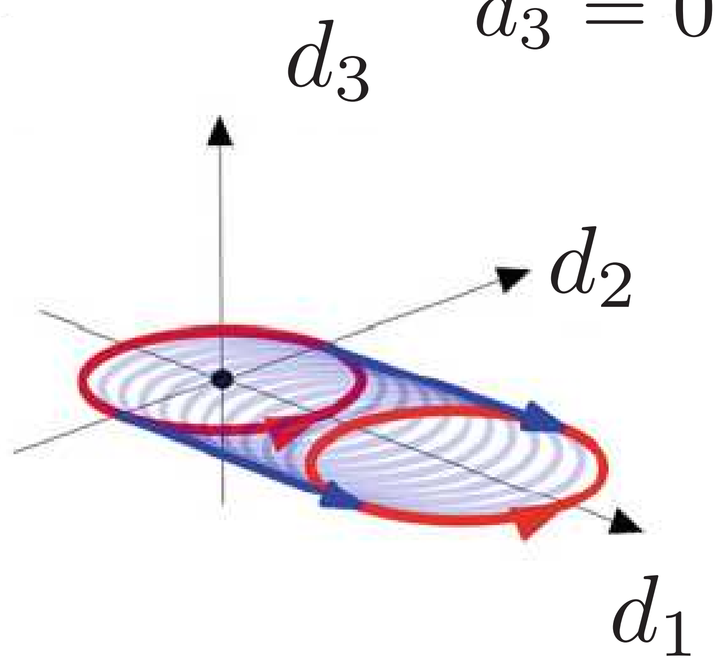


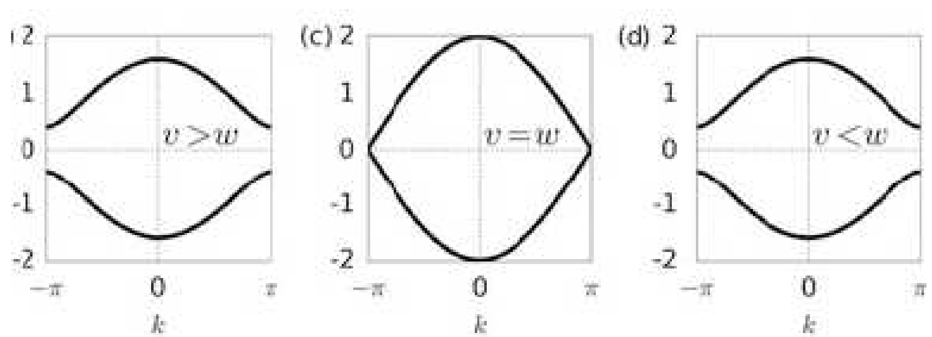
We cannot deform the loop and change the winding number without crossing the origin (closing the gap) and keeping the chiral symmetry

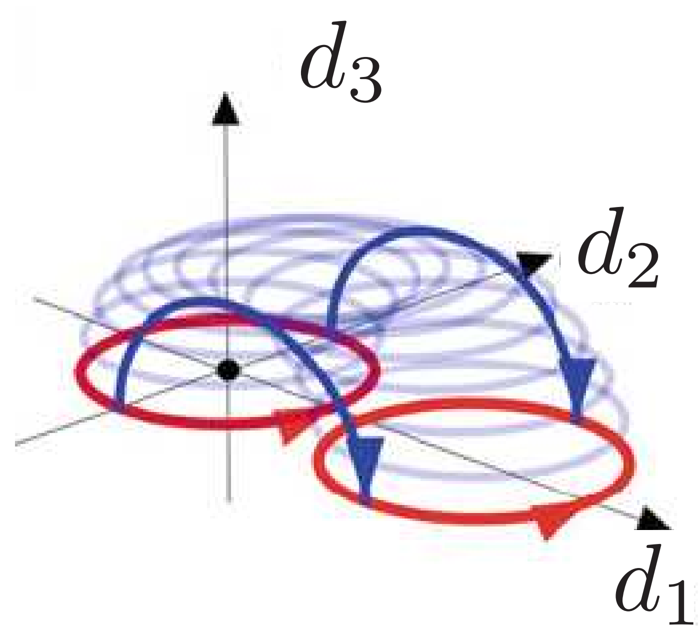
$$H \to H + \Delta\sigma_3$$

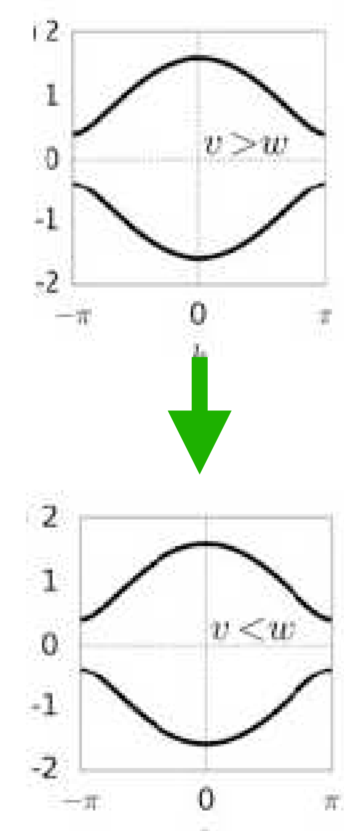
$$d_3 = 0$$

$$U = \sigma_3 \qquad HU \neq -UH$$



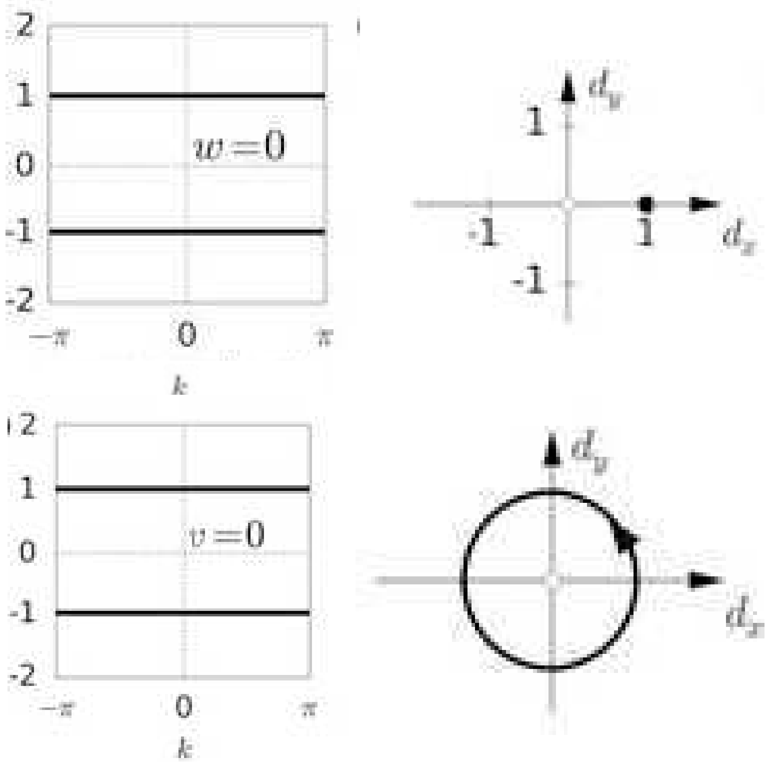


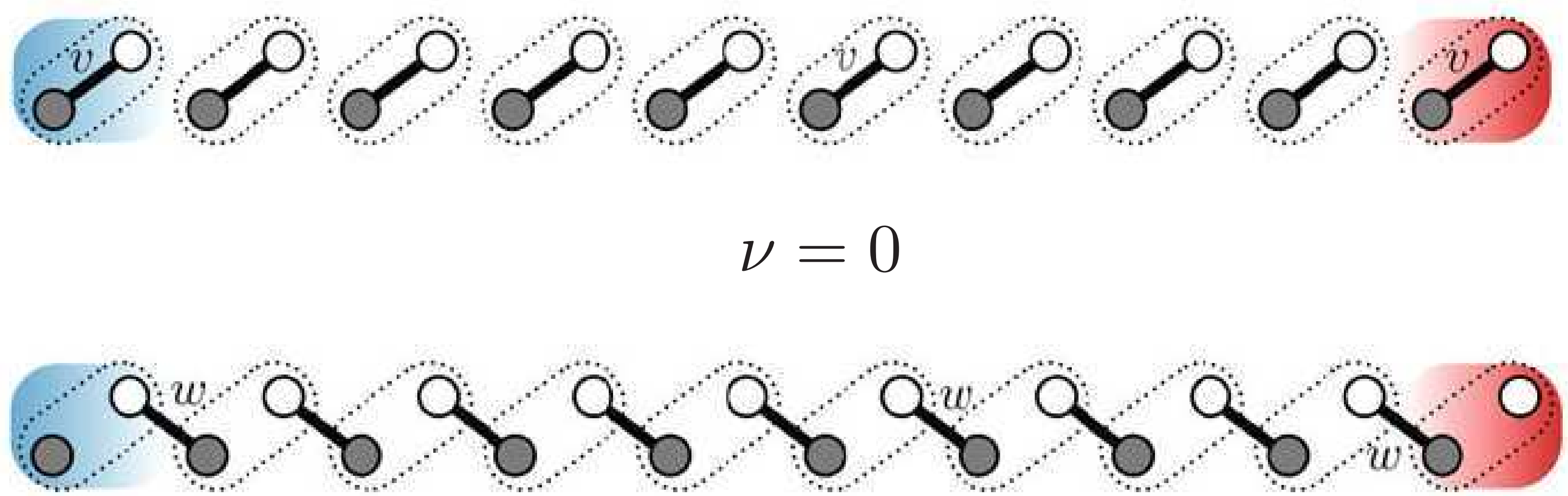




II. Definition of symmetry protected topological phases

What are the consequences of all of this? Surface/edge states





$\nu = 0$

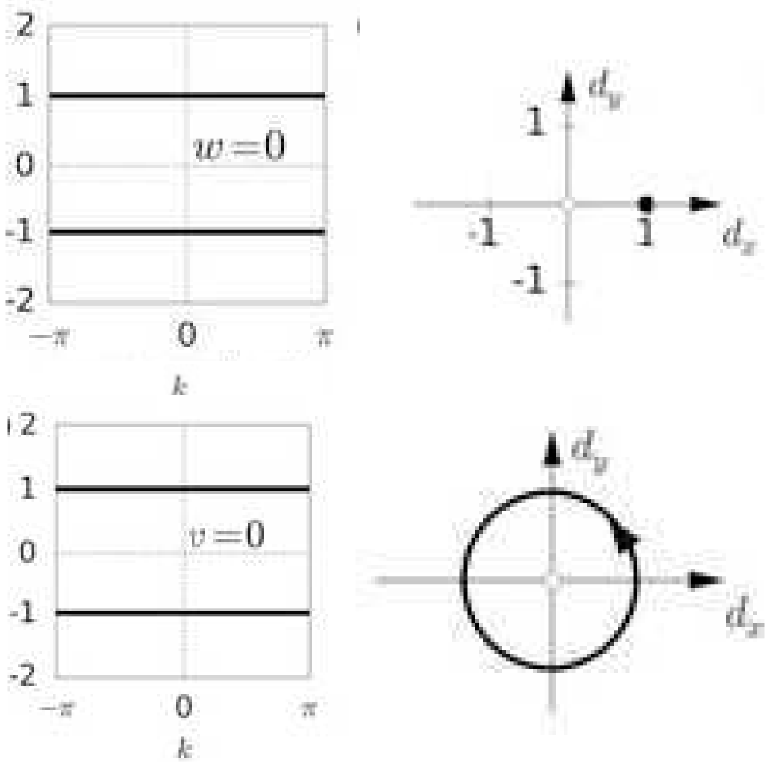
$\nu = 1$

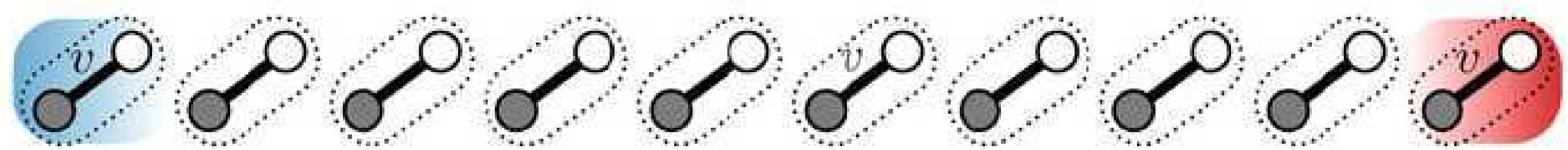
$$H = \sum_n v c^+_{A,n} c_{B,n} + w c^+_{A,n+1} c_{b,n} + h.c.$$

$$H = \begin{pmatrix} 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & v \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \end{pmatrix}$$

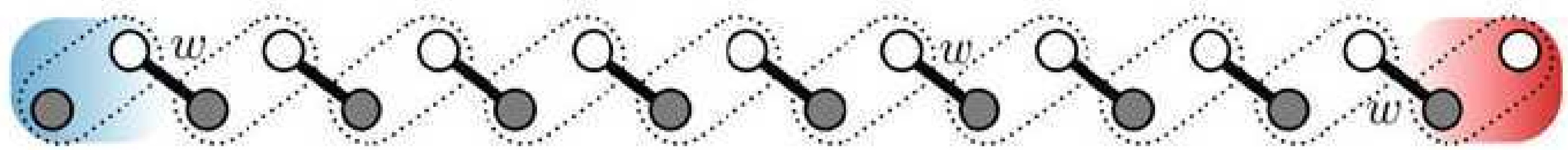
II. Definition of symmetry protected topological phases

What are the consequences of all of this? Surface/edge states





$$\nu = 0$$



$$\nu = 1$$

$$H = \sum_n v c^{+}_{A,n} c_{B,n} + w c^{+}_{A,n+1} c_{b,n} + h.c.$$

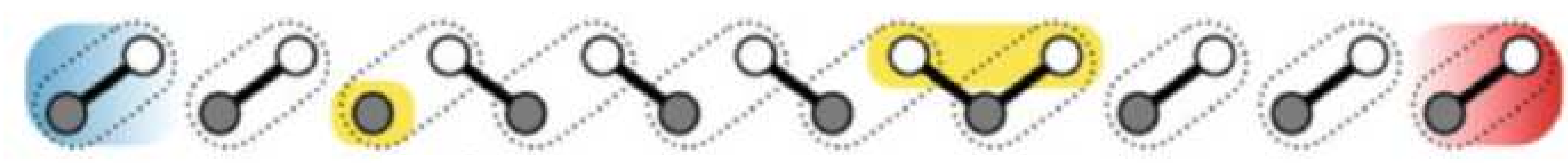
$$H\,|1, A\rangle = 0$$

$$H\,|N, B\rangle = 0$$

edge states
(one per side!)

II. Definition of symmetry protected topological phases

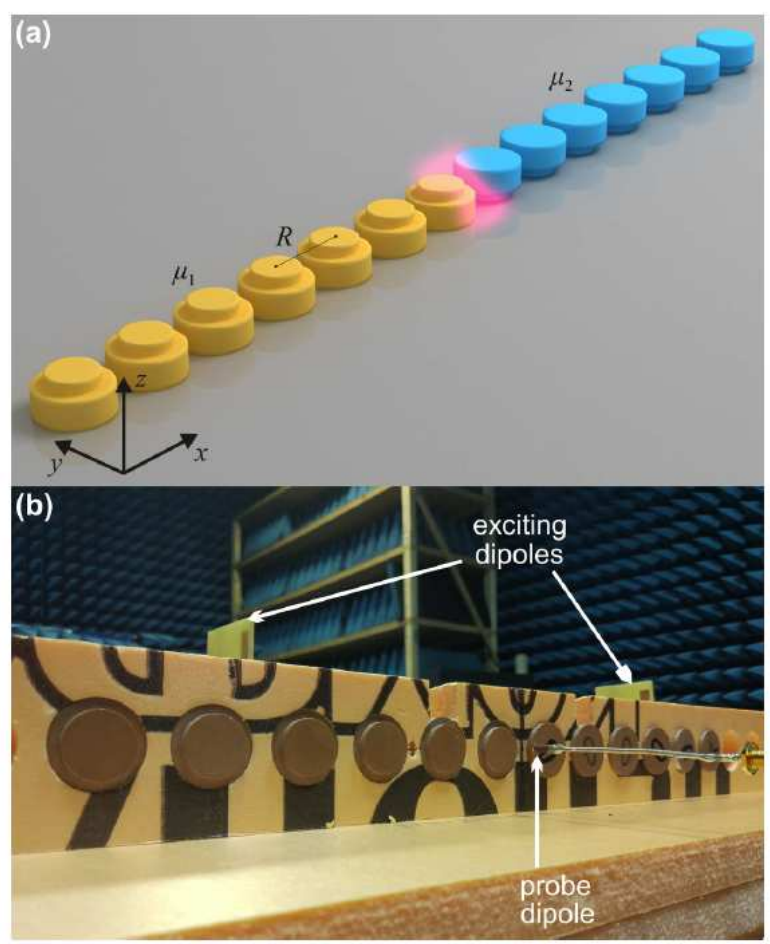
What if we glue a trivial with a nontrivial phase?

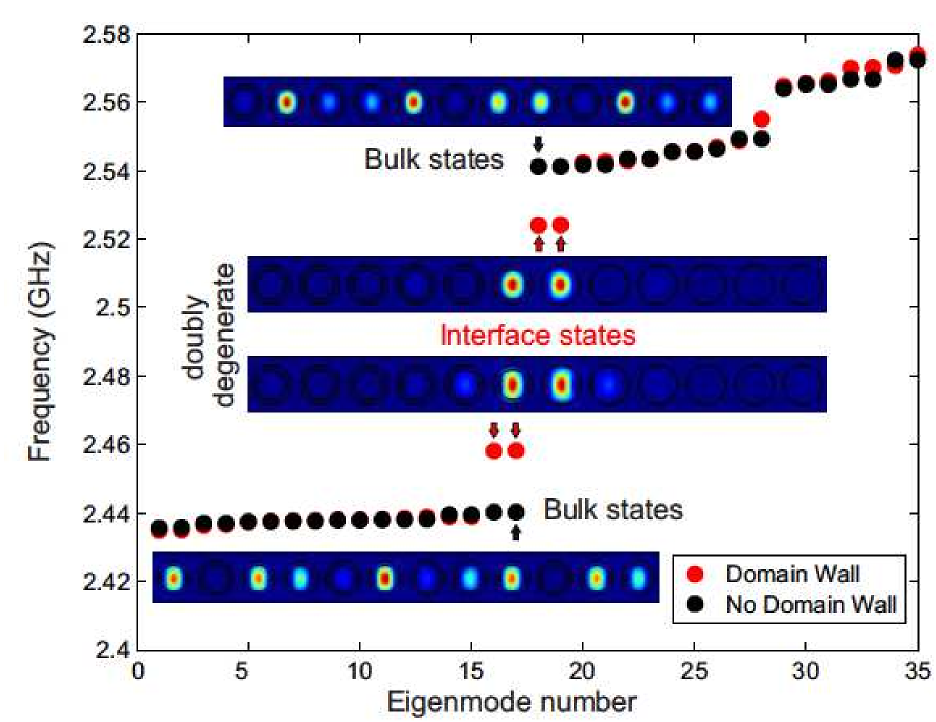


$\nu = 1$

$\nu = 0$

A zero energy state Marks the boundary Between trivial and nontrivial Insulators (domain wall state)





A Gorlach et al PRB 99 205122 (2019)

III. Berry phase: "The mother of the lamb"

We have seen in the SSH model that one way to construct the topological invariant is By using the eigenstates

$$\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} i \langle s| \partial_k |s\rangle \, dk \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{A}_k dk$$

$\mathcal{A}_k = i\langle s|\partial_k s\rangle$ Berry connection $\quad |s\rangle \to |s'\rangle = e^{i\phi(k)}|s\rangle$

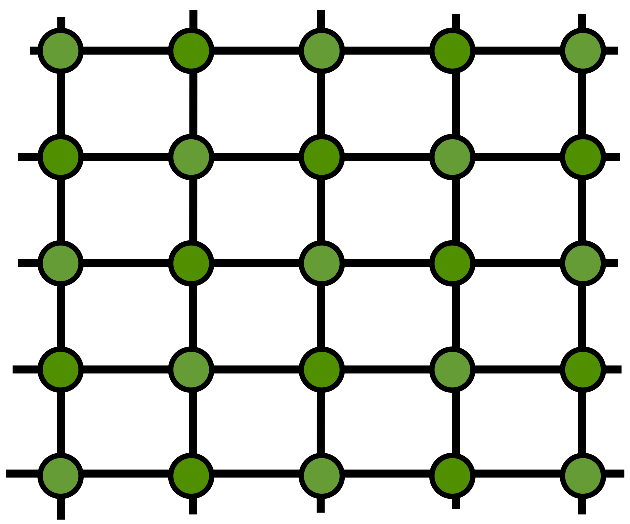
$$\mathcal{A}'_k = i\langle s'|\partial_k s'\rangle = i\langle s|\partial_k s\rangle - \partial_k \phi(k)$$

After a local change (in momentum space) of the phase of the state, the quantity $\mathcal{A}_k$ changes as a connection

As long as $\phi(k)$ is not singular $\quad \Delta\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} \partial\phi(k) = 0$

III. Berry phase: "The mother of the lamb"

Let's move on to two dimensions

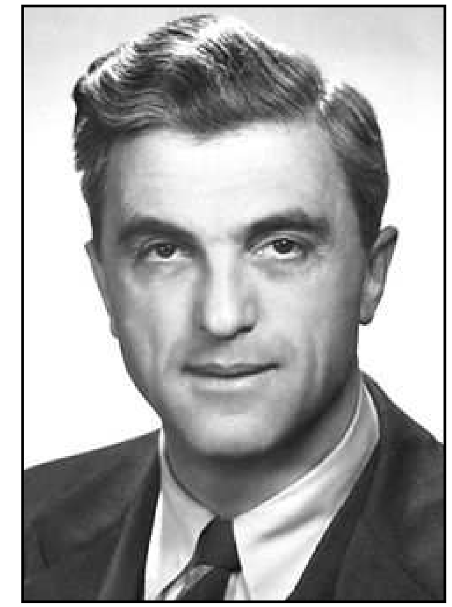


$$H = H_{\boldsymbol{k}} + V$$

$$V(\boldsymbol{r}) = V(\boldsymbol{r} + \boldsymbol{a})$$

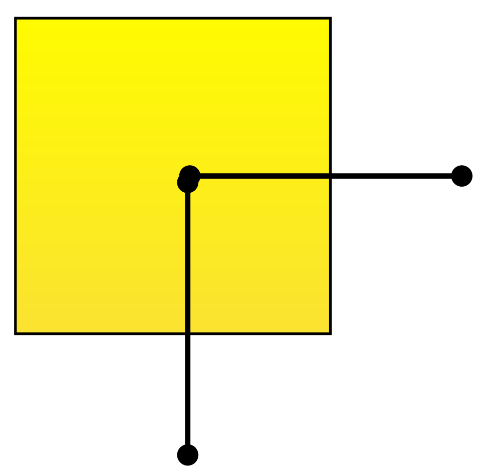
$$\psi = e^{i\boldsymbol{k}\boldsymbol{r}} u_{\boldsymbol{k}}(\boldsymbol{r})$$

$$u_{\boldsymbol{k}}(\boldsymbol{r} + \boldsymbol{a}) = e^{i\boldsymbol{k}\boldsymbol{a}} u_{\boldsymbol{k}}(\boldsymbol{r})$$

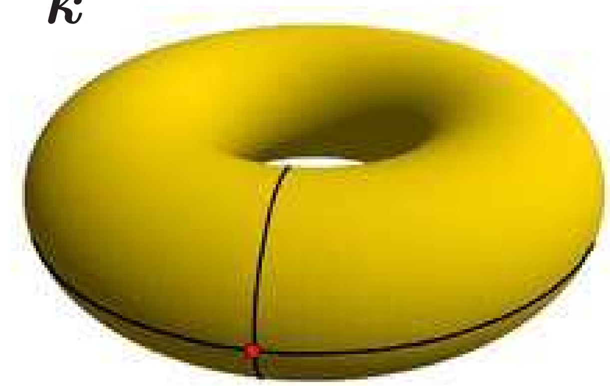


F Bloch (1905-1983)

$$\varepsilon_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}+\boldsymbol{G}}$$



$\boldsymbol{k}$

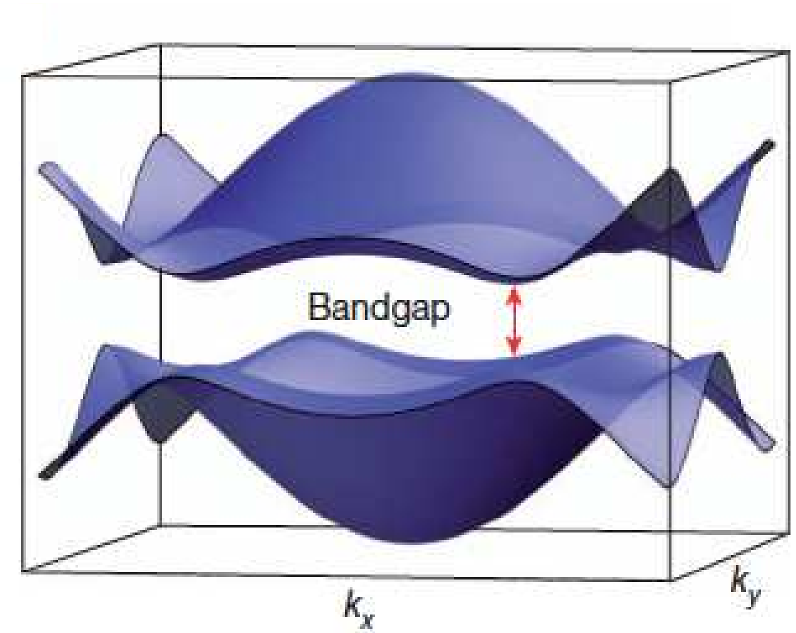


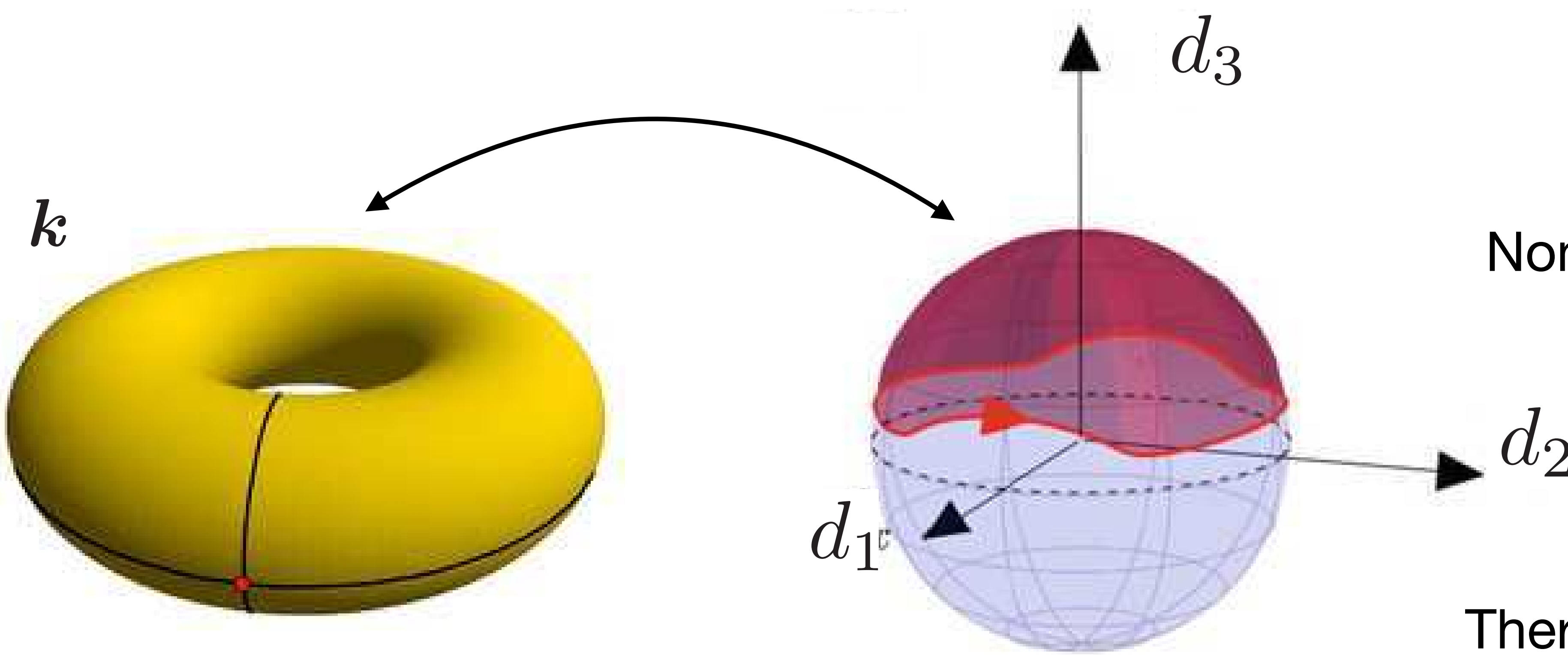
III. Berry phase:"The mother of the lamb"

Let's move on to two dimensions

$$H(\boldsymbol{k}) = \tau v_F \sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 h(\boldsymbol{k})$$

$$\boldsymbol{d_k} = (k_1, k_2, h(\boldsymbol{k}))$$





Non trivial Homotopy group

There is a generalization of
A winding number

III. Berry phase:"The mother of the lamb"

Let's move on to two dimensions

$$H(\boldsymbol{k}) = \tau v_F \sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 h(\boldsymbol{k})$$

$$\boldsymbol{\mathcal{A}}(\boldsymbol{k}) = i \langle s | \partial_{\boldsymbol{k}} s \rangle$$

$$\boldsymbol{n_k} = \frac{\boldsymbol{d_k}}{|\boldsymbol{d_k}|}$$

Pontryagin index

$$\nu = \frac{1}{4\pi} \oint_{\mathcal{M}} \boldsymbol{\mathcal{A}}(\boldsymbol{k}) \cdot d\boldsymbol{k} = \frac{1}{4\pi} \int \boldsymbol{n_k} \cdot (\partial_1 \boldsymbol{n_k} \times \partial_2 \boldsymbol{n_k}) d^2 k$$

$$\nu = \frac{1}{4\pi} \int_{\boldsymbol{M}} \boldsymbol{\partial_k} \times \boldsymbol{\mathcal{A}}(\boldsymbol{k}) d^2 \boldsymbol{k}$$

Usually known as Chern number

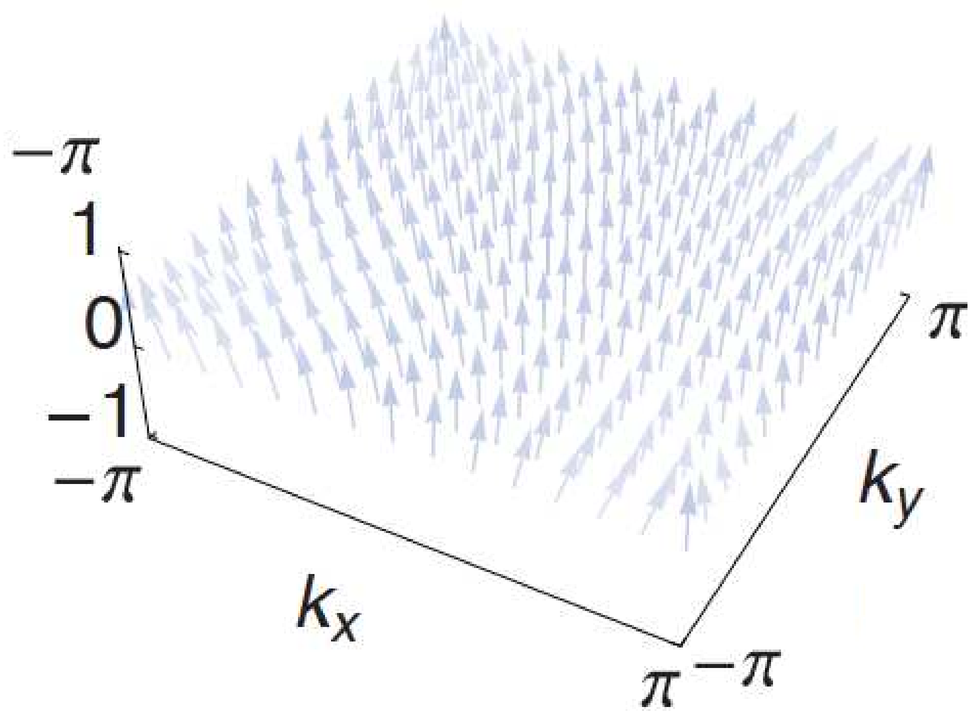
Berry curvature $\boldsymbol{\Omega}(\boldsymbol{k})$

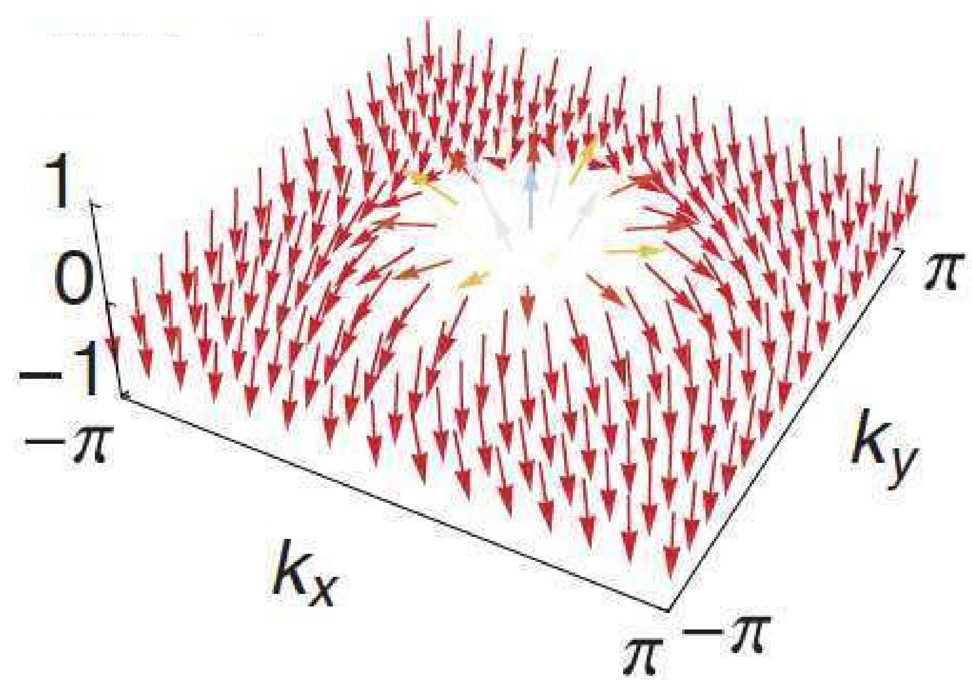
III. Berry phase: "The mother of the lamb"

Let's move on to two dimensions

$$H(\boldsymbol{k}) = \tau v_F \sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 h(\boldsymbol{k})$$

$$\nu = \frac{1}{4\pi} \int_{\mathcal{M}} d^2\boldsymbol{k} \cdot \boldsymbol{\Omega}(\boldsymbol{k})$$





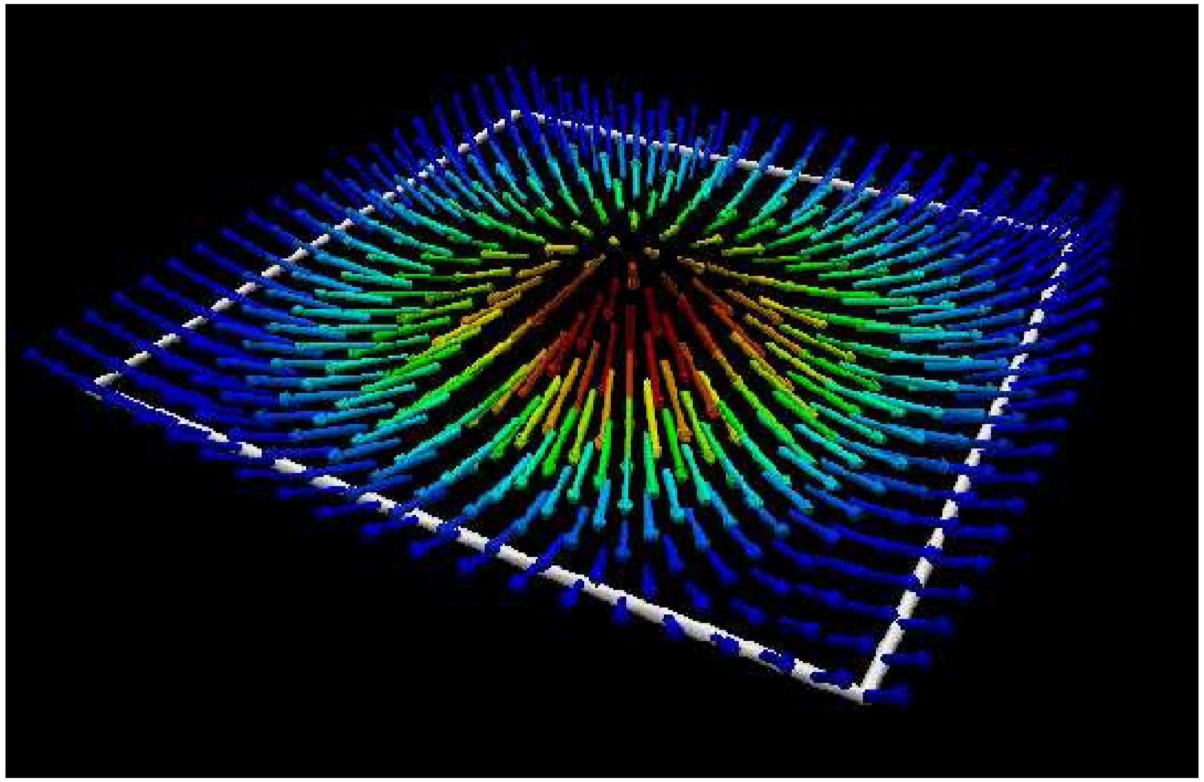
Number of times the vector $\boldsymbol{\Omega}(\boldsymbol{k})$ winds

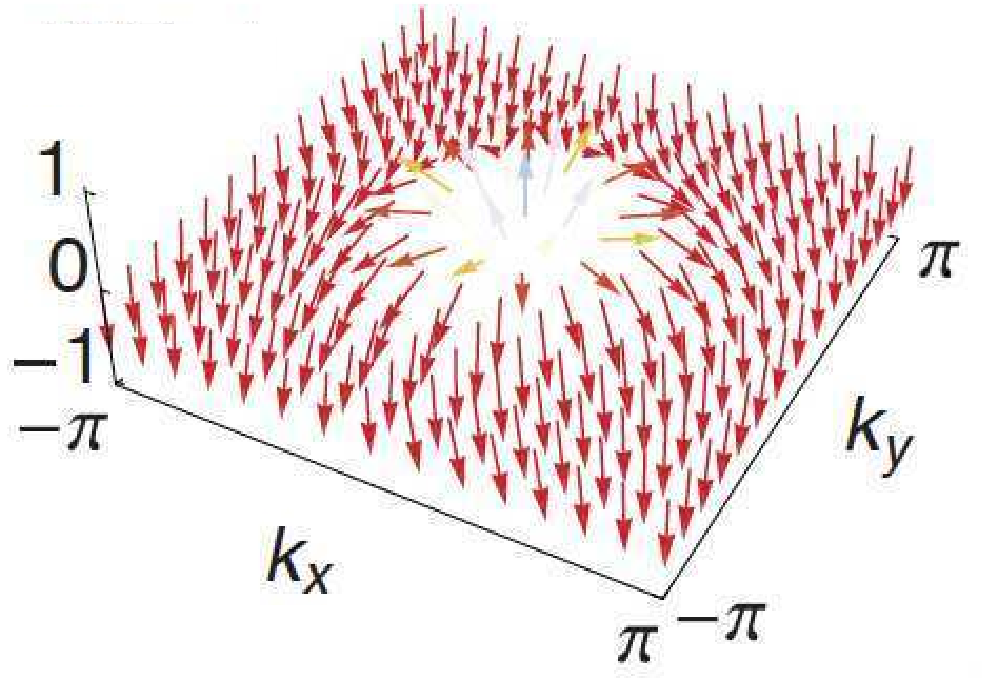
III. Berry phase:"The mother of the lamb"

Let's move on to two dimensions

$$H(\boldsymbol{k}) = \tau v_F \sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 h(\boldsymbol{k})$$

$$\nu = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\boldsymbol{k} \cdot \boldsymbol{\Omega}(\boldsymbol{k})$$





Number of times the vector $\boldsymbol{\Omega}(\boldsymbol{k})$ winds

III. Berry phase:"The mother of the lamb"

How does it change under time reversal?

$$\boldsymbol{k} \to -\boldsymbol{k} \qquad u(\boldsymbol{k},\boldsymbol{r}) \to u^*(-\boldsymbol{k},\boldsymbol{r})$$

$$\boldsymbol{\Omega}(\boldsymbol{k}) = \boldsymbol{\partial}_{\boldsymbol{k}} \times \boldsymbol{\mathcal{A}}(\boldsymbol{k})$$

$$\int d\boldsymbol{r} u(\boldsymbol{k},\boldsymbol{r})\partial_i u^*(\boldsymbol{k},\boldsymbol{r}) = -\int d\boldsymbol{r} u^*(\boldsymbol{k},\boldsymbol{r})\partial_i u(\boldsymbol{k},\boldsymbol{r})$$

$$\boldsymbol{\mathcal{A}}(\boldsymbol{k}) = i\int d\boldsymbol{r} u^*(\boldsymbol{k},\boldsymbol{r})\frac{\partial}{\partial k_i}u(\boldsymbol{k},\boldsymbol{r})$$

$$\boldsymbol{\mathcal{A}}^{\mathcal{T}}(\boldsymbol{k}) = \boldsymbol{\mathcal{A}}(-\boldsymbol{k})$$

$$\boldsymbol{\Omega}^{\mathcal{T}}(\boldsymbol{k}) = -\boldsymbol{\Omega}(-\boldsymbol{k})$$ It's an odd function of $\boldsymbol{k}$

Berry connection transforms
as the position under TRS (and inversion)

If the system is time
reversal symmetric

$$\boldsymbol{\Omega}(\boldsymbol{k}) = -\boldsymbol{\Omega}(-\boldsymbol{k})$$

$$\boxed{\nu = \int d^2\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{k}) = 0}$$

III. Berry phase: "The mother of the lamb"

Electromagnetism in real space

U(1) local invariance of the wave function

$$\psi(r) \rightarrow e^{i\phi(r)}\psi(r)$$

and

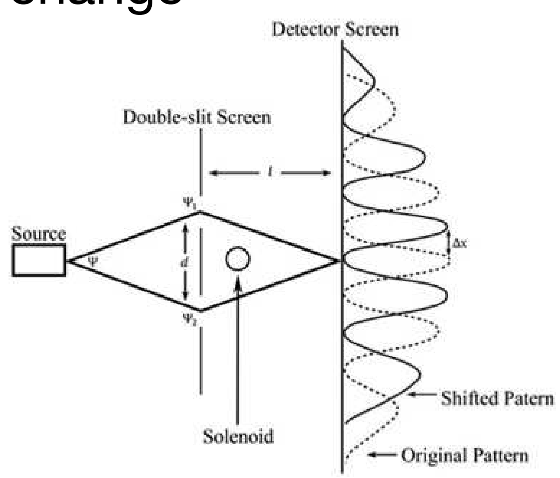
$$\hat{k}_\mu = i\partial_\mu \rightarrow \hat{k}_\mu - A_\mu$$

Any field configuration such

$A_\mu = A_\mu - \partial_\mu \phi(r)$ has no physical effect.

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ The magnetic field does not change

Aharonov-Bohm effect



Berry phase in momentum space

U(1) local invariance of the Bloch w.f. in momentum space

$$u(r,k) \rightarrow e^{i\phi(k)}u(r,k)$$

and

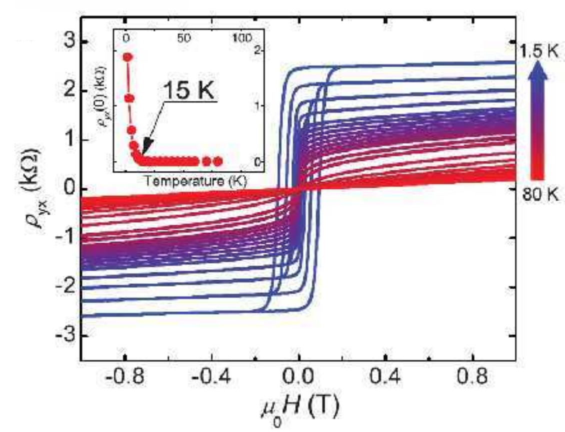
$$\hat{r}_i = i\partial_{k_i} \rightarrow \hat{r}_i - \mathcal{A}_i$$

Any Berry connection configuration such

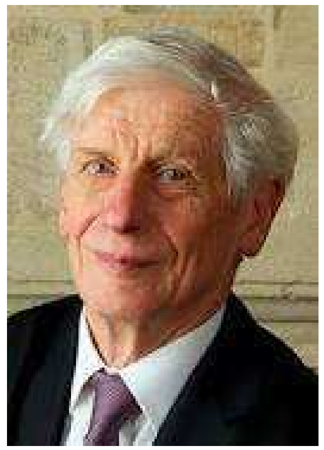
$\mathcal{A}_i = \mathcal{A}_i - \partial_i \phi(k)$ has no physical effect.

$\boldsymbol{\Omega} = \boldsymbol{\nabla}_{\boldsymbol{k}} \times \boldsymbol{\mathcal{A}}$ Berry curvature does not change

QAHE

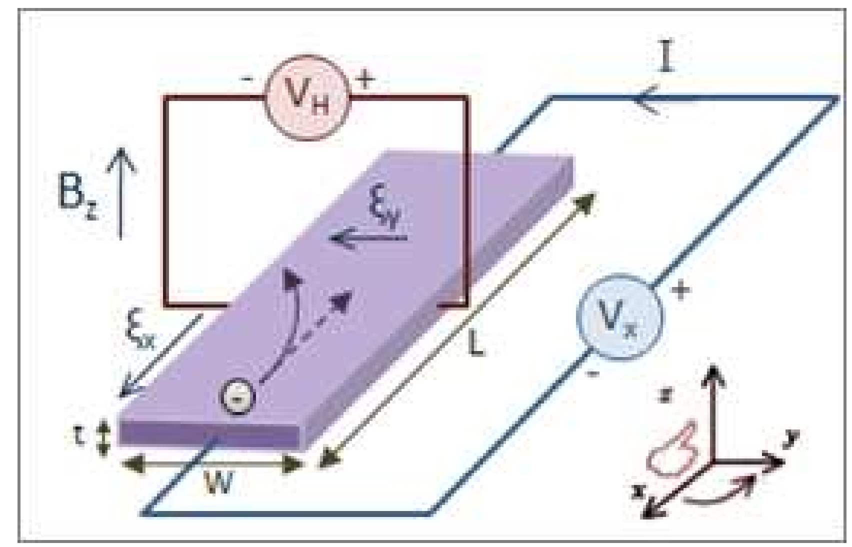


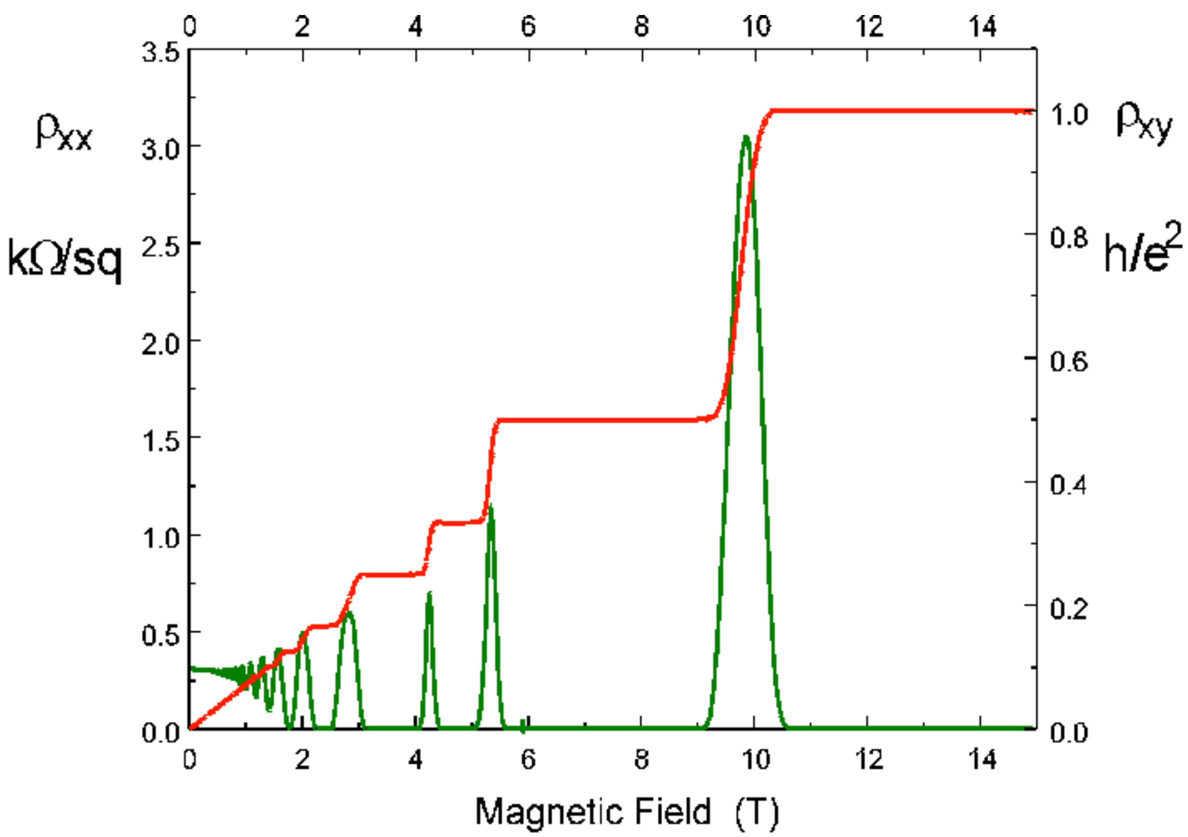
III. Berry phase:"The mother of the lamb"



Can the topological invariants be measured?

Quantum Hall effect

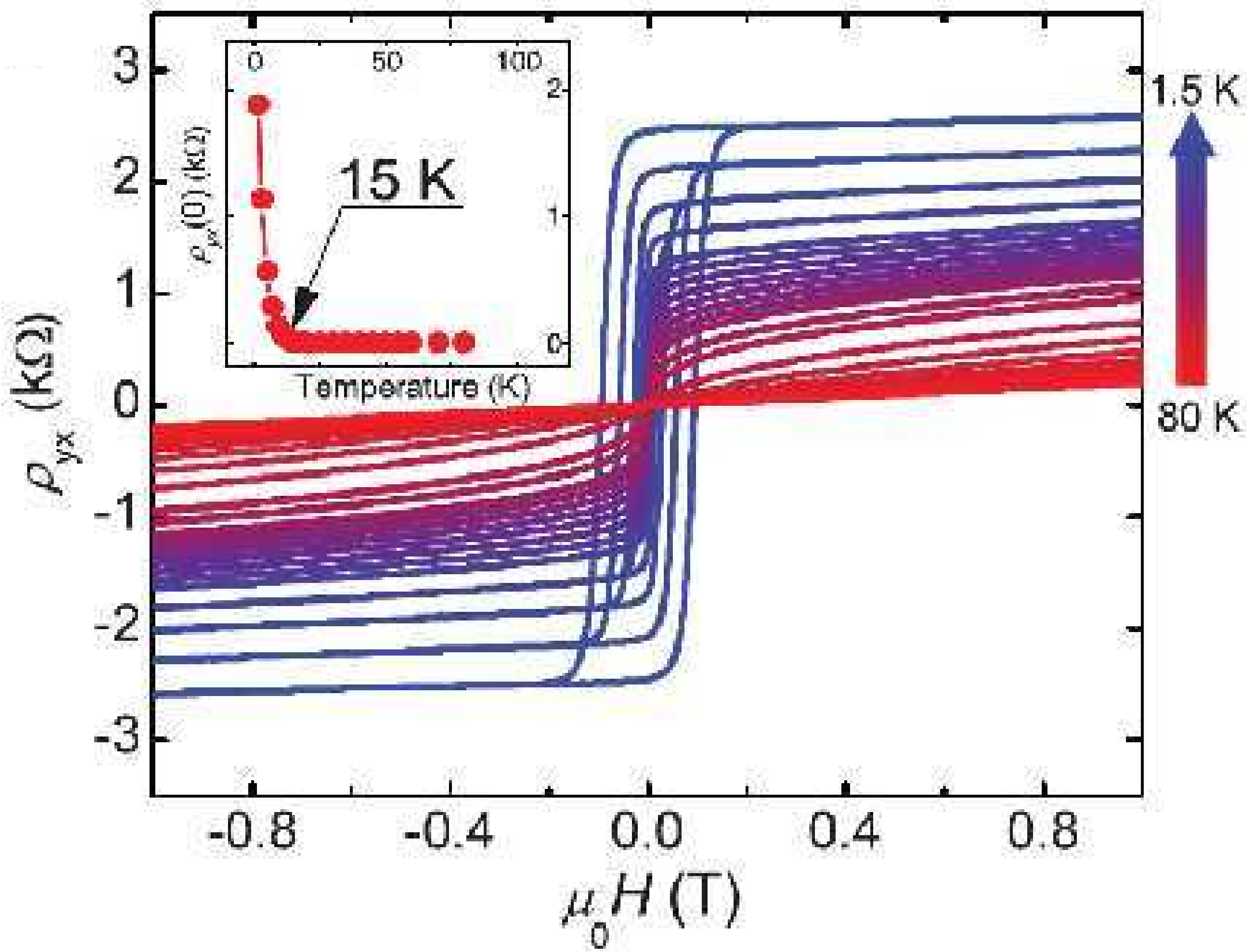




$$\sigma_H = \frac{e^2}{h} N$$

III. Berry phase: "The mother of the lamb"

QAHE:



@ B=0

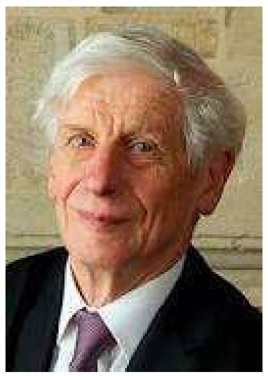
$$\rho_{xy} \neq 0$$

$$\sigma_H = \frac{e^2}{h} N$$

magnetically doped
3DTI
with ordered magnetization

III. Berry phase: "The mother of the lamb"

Hall conductance as integral of the Berry curvature



$$H = H(\boldsymbol{k}) + V \qquad V = e\boldsymbol{E}\cdot\boldsymbol{r} = -ie\boldsymbol{E}\cdot\partial_{\boldsymbol{k}}$$

$$H(\boldsymbol{k})\,|u_n(\boldsymbol{k})\rangle_0 = E_n(\boldsymbol{k})\,|u_n(\boldsymbol{k})\rangle_0 \qquad \boldsymbol{J} = e\boldsymbol{v} = e\frac{d\boldsymbol{r}}{dt} = \frac{e}{i\hbar}[H,\boldsymbol{r}] = \frac{e}{\hbar}\frac{\partial H}{\partial \boldsymbol{k}}$$

$$|\tilde{u}_n(\boldsymbol{k})\rangle = |u_n(\boldsymbol{k})\rangle - ieE_i \sum_{m\neq n} |u_m(\boldsymbol{k})\rangle \frac{\langle u_m(\boldsymbol{k})|\,\partial_{k_i}\,|u_n(\boldsymbol{k})\rangle}{E_m - E_n}$$

$$\langle\tilde{u}_n(\boldsymbol{k})| = \langle u_n(\boldsymbol{k})| + ieE_i \sum_{m\neq n} \frac{\langle u_n(\boldsymbol{k})|\,\partial_{k_i}\,|u_m(\boldsymbol{k})\rangle}{E_m - E_n} \langle u_m(\boldsymbol{k})|$$

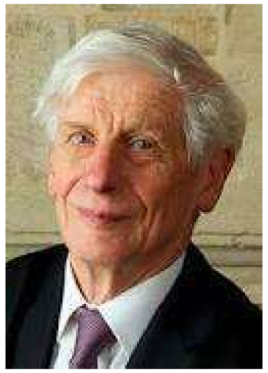
$$\langle \boldsymbol{J}\rangle = \frac{e}{4\pi^2\hbar}\int d^2\boldsymbol{k}\,\langle\tilde{u}_n|\,\frac{\partial H}{\partial \boldsymbol{k}}\,|\tilde{u}_n\rangle$$

first order perturbation theory!

DJ Thouless, M Kohmoto, MP Nightingale, M Den Nijs, PRL, 49, 405 (1982)

III. Berry phase: "The mother of the lamb"

Hall conductance as integral of the Berry curvature



$$H|n\rangle = \varepsilon_n |n\rangle \qquad \langle m| H = \langle m| \varepsilon_m \qquad n \neq m \qquad \langle n|m\rangle = 0$$

$$\partial_i H|n\rangle + H|\partial_i n\rangle = \partial_i \varepsilon_n |n\rangle + \varepsilon_n |\partial_i n\rangle$$

$\langle m| \times$

$$\left\langle m \middle| \frac{\partial H}{\partial k_i} \middle| n \right\rangle = (\varepsilon_n - \varepsilon_m)\langle m|\partial_i n\rangle$$

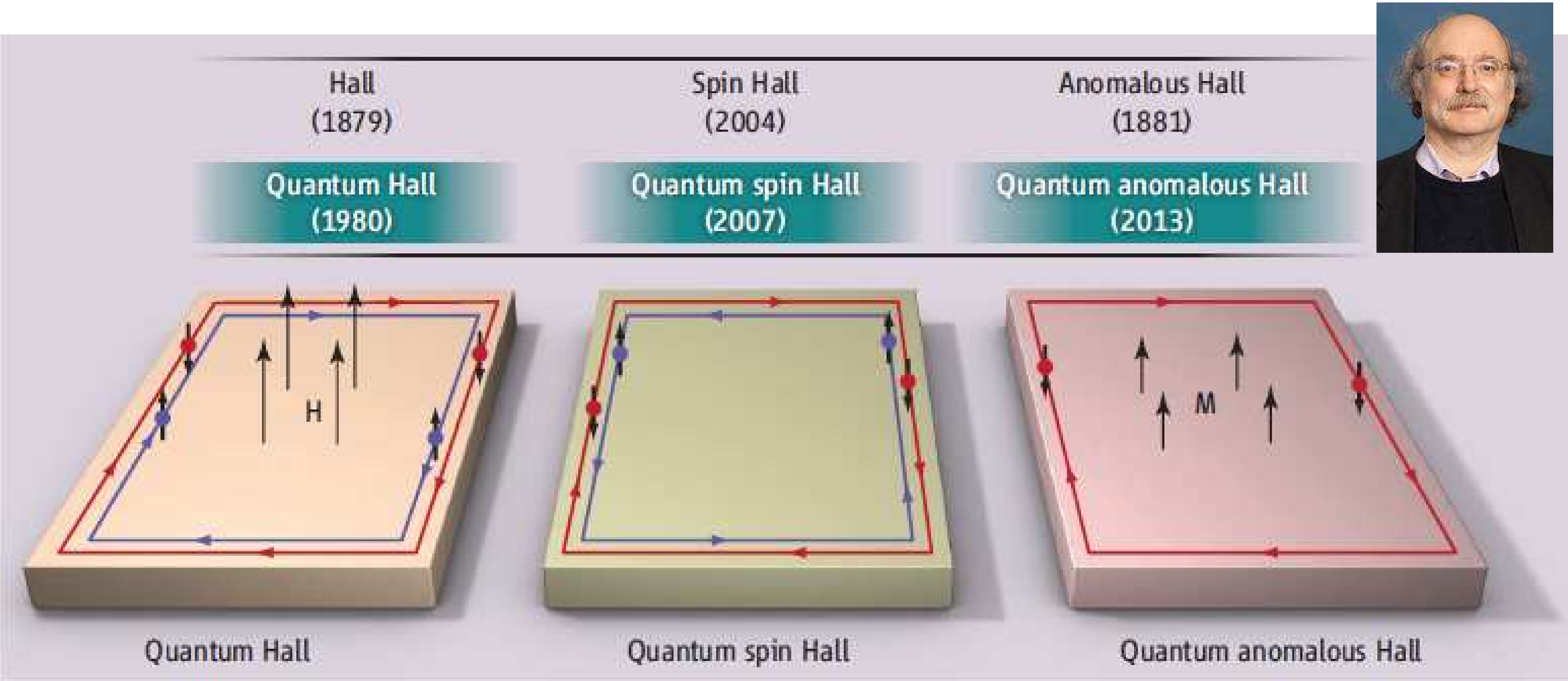
$$\langle J_j\rangle = i\frac{e^2}{4\pi^2}E_i \int d^2\boldsymbol{k} \sum_{m\neq n} \langle u_m(\boldsymbol{k})|\,\partial_{k_j}\,|u_n(\boldsymbol{k})\rangle \langle u_n(\boldsymbol{k})|\,\partial_{k_i}\,|u_m(\boldsymbol{k})\rangle -$$
$$-i\frac{e^2}{4\pi^2}E_i \int d^2\boldsymbol{k} \sum_{m\neq n} \langle u_n(\boldsymbol{k})|\,\partial_{k_j}\,|u_m(\boldsymbol{k})\rangle \langle u_m(\boldsymbol{k})|\,\partial_{k_i}\,|u_n(\boldsymbol{k})\rangle$$

$$\langle J_i\rangle = \frac{e^2}{2\pi h}E_j \int d^2\boldsymbol{k}\,\partial_{k_i} i\,\langle u_n(\boldsymbol{k})|\partial_{k_j} u_n(\boldsymbol{k})\rangle$$

$$\sigma_{12} = \frac{e^2}{2\pi h}\int d^2\boldsymbol{k}\,\partial_{\boldsymbol{k}} \times \boldsymbol{\mathcal{A}}(\boldsymbol{k}) \equiv \frac{e^2}{2\pi h}\int d^2\boldsymbol{k}\,\Omega(\boldsymbol{k})$$

DJ Thouless, M Kohmoto, MP Nightingale, M Den Nijs, PRL, 49, 405 (1982)

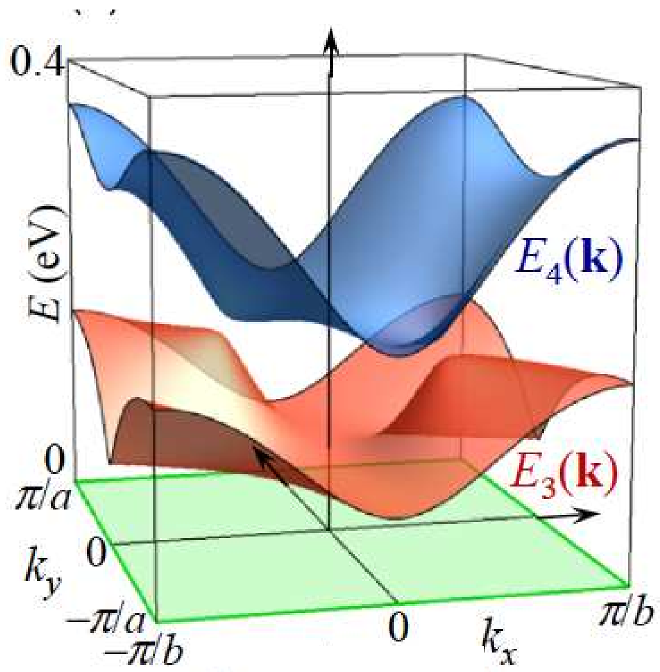
III. Berry phase:“The mother of the lamb”



This one is time reversal invariant

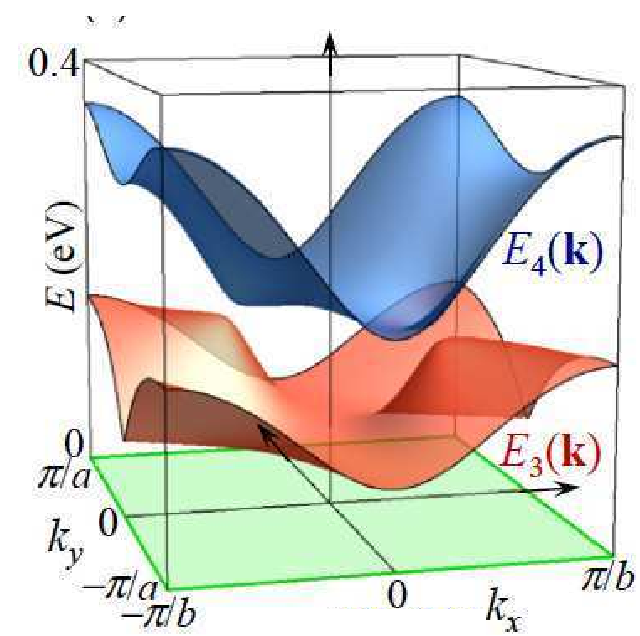
IV. Time reversal invariant topological insulators

We have seen that TRS constraints the Berry curv. to be odd in momentum space.

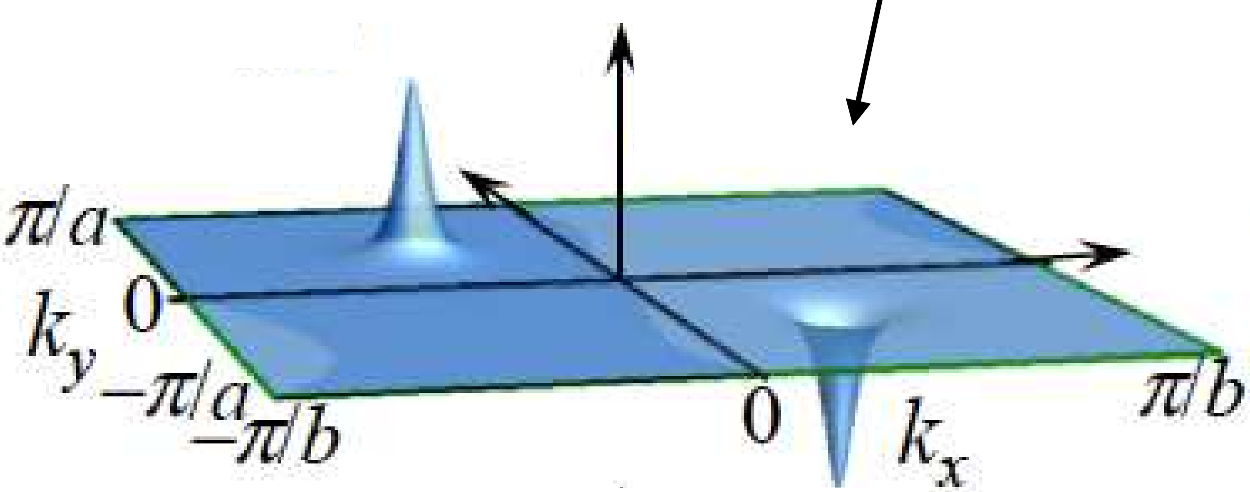


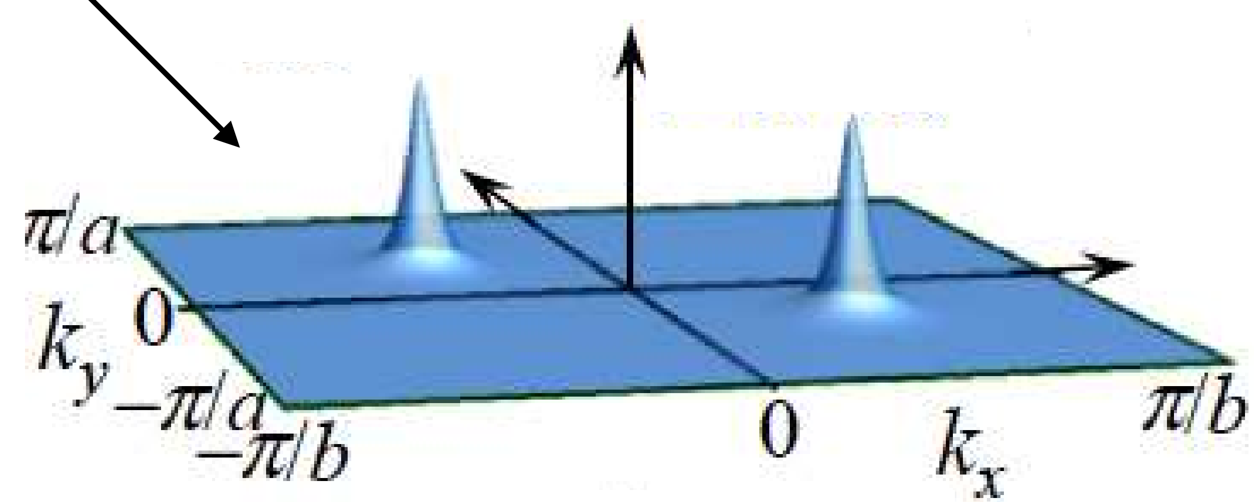
$$\boldsymbol{\Omega}(\boldsymbol{k}) = -\boldsymbol{\Omega}(-\boldsymbol{k})$$

$$\int d^2\boldsymbol{k} \cdot \boldsymbol{\Omega}_n = 0$$



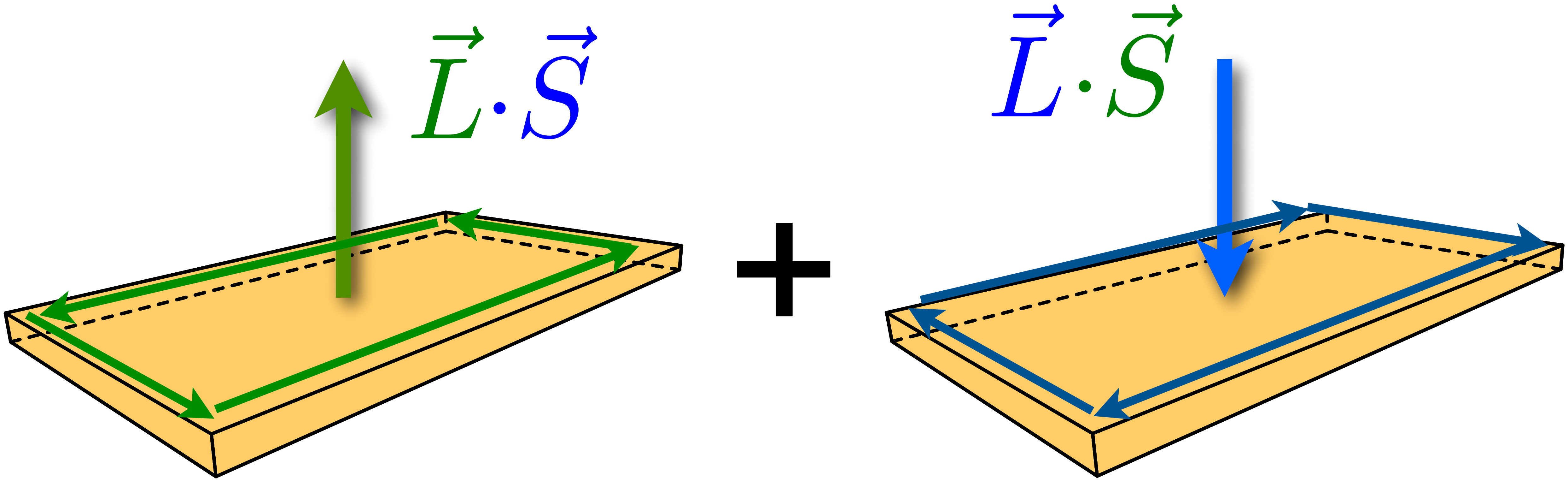
$$\int d^2\boldsymbol{k} \cdot \boldsymbol{\Omega}_n = \mathcal{C}$$





IV. Time reversal invariant topological insulators

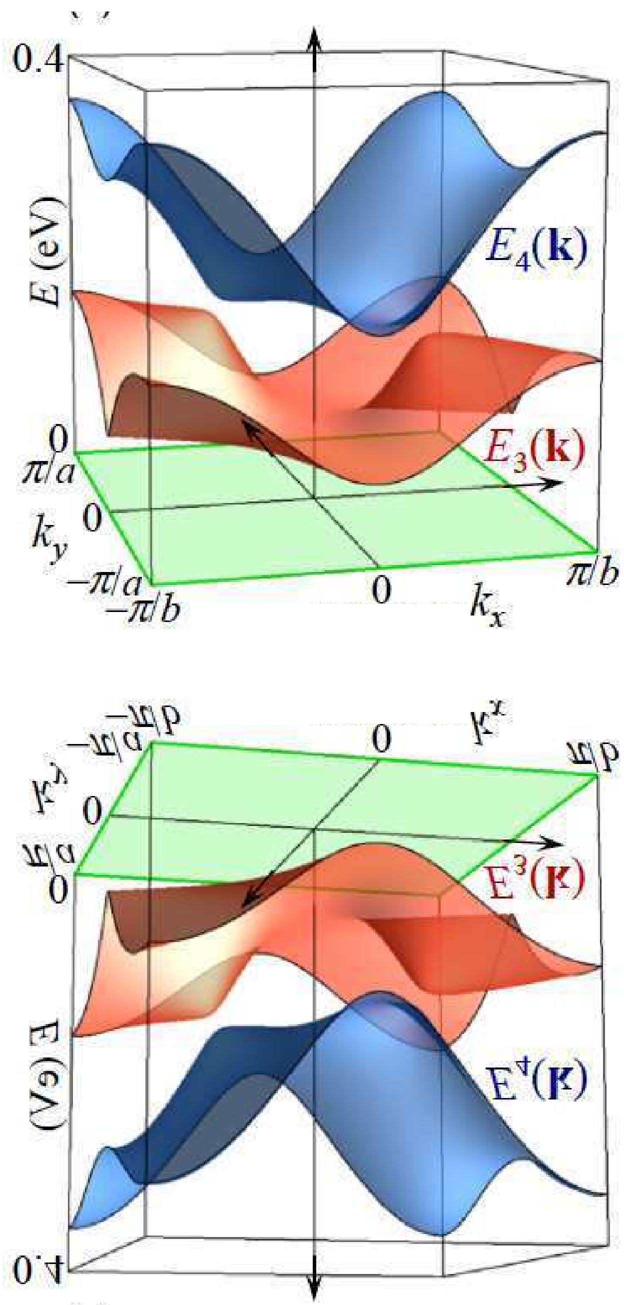
But we can do other things

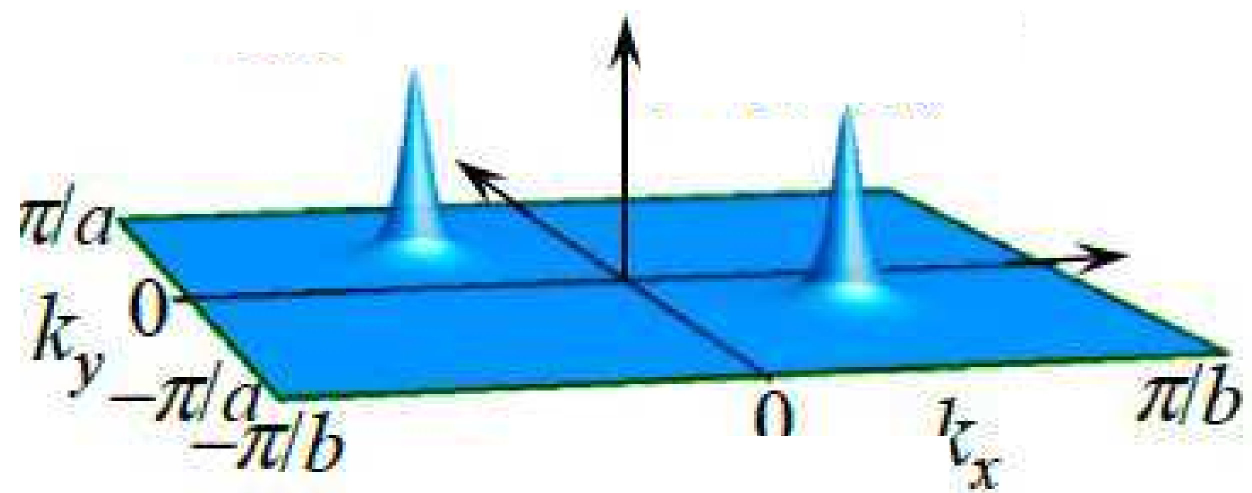


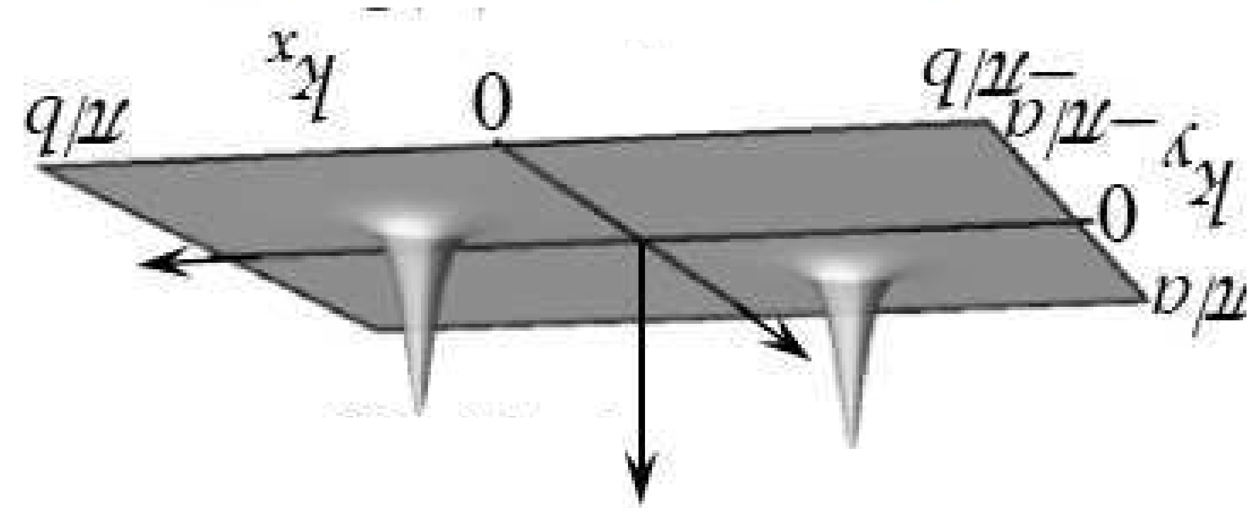
We have two copies of the Haldane model (local orbital magnetic effects with zero net magnetization) one per spin projection

IV. Time reversal invariant topological insulators

We have two copies of the Haldane model (local orbital magnetic effects with zero net magnetization) one per spin projection



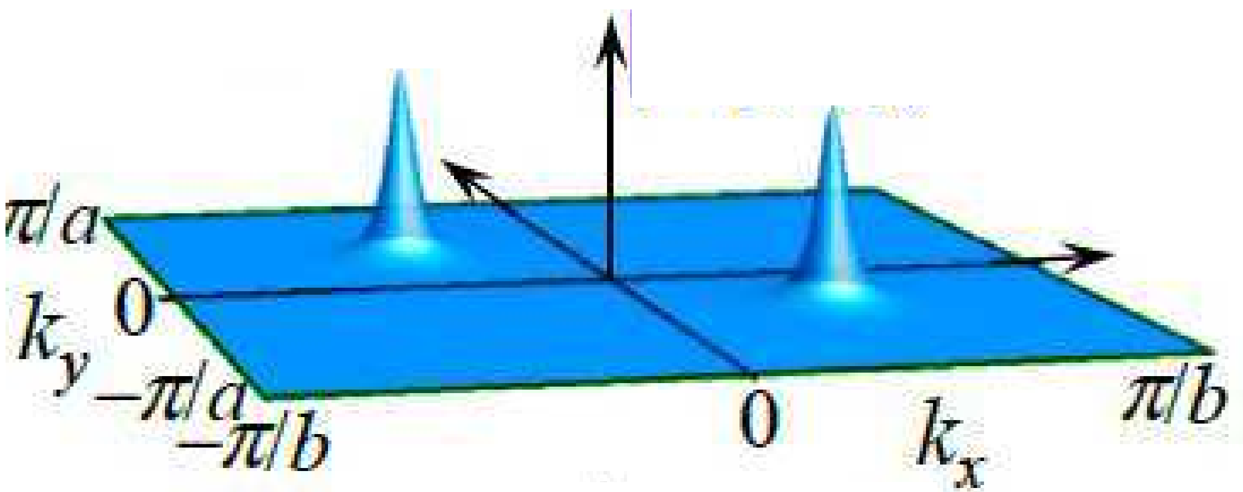


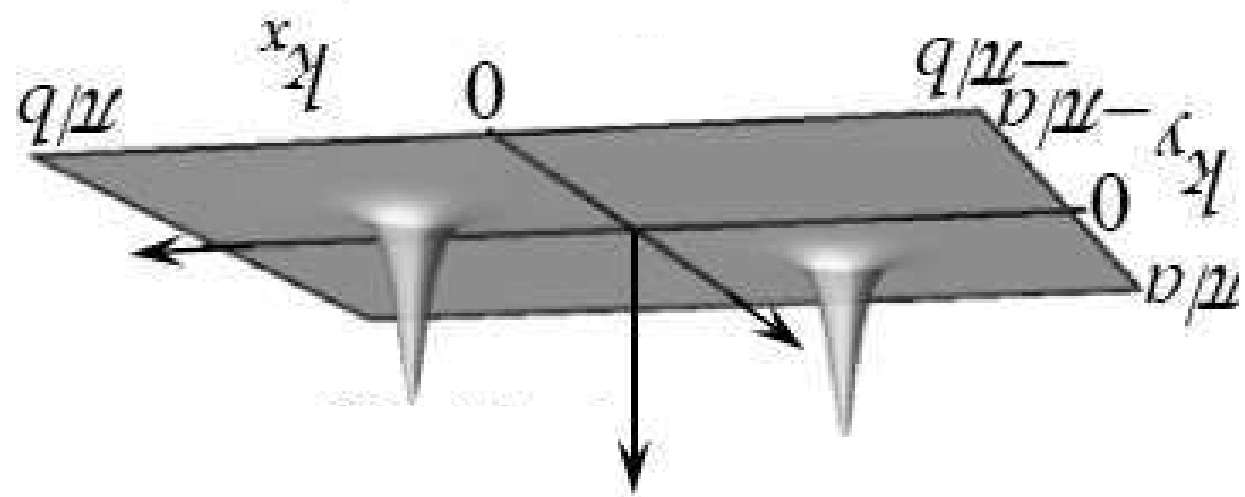


$$\int d^2\boldsymbol{k}\cdot\boldsymbol{\Omega} = \mathcal{C} \qquad\qquad \int d^2\boldsymbol{k}\cdot\boldsymbol{\Omega} = -\mathcal{C}$$

IV. Time reversal invariant topological insulators

We have two copies of the Haldane model (local orbital magnetic effects with zero net magnetization) one per spin projection





$$\int d^2\boldsymbol{k}\cdot\boldsymbol{\Omega} = \mathcal{C}$$

$$\int d^2\boldsymbol{k}\cdot\boldsymbol{\Omega} = -\mathcal{C}$$

$$\mathcal{C}_T = \mathcal{C} - \mathcal{C} = 0$$

$$\mathcal{C}_S = \mathcal{C} - \mathcal{C} \neq 0$$

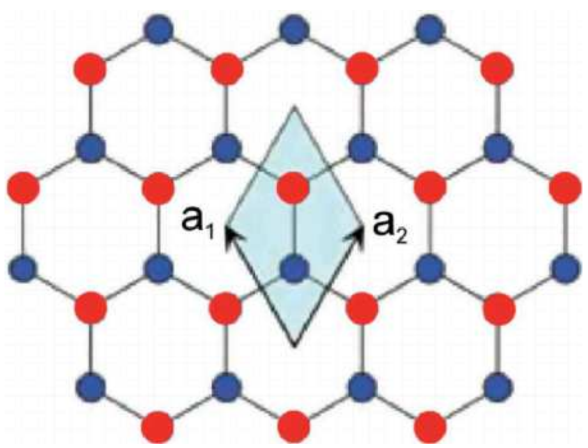
As a SPT phase, it can be stabilised with TRS (spinful T is antiunitary)
Or other discrete symmetry: mirror symmetry, particle-hole symm,…

V. Topological insulators in three dimensions

How these things work in 3D?

$$H = v\boldsymbol{\alpha}\cdot\boldsymbol{k} + m(\boldsymbol{k})\beta$$

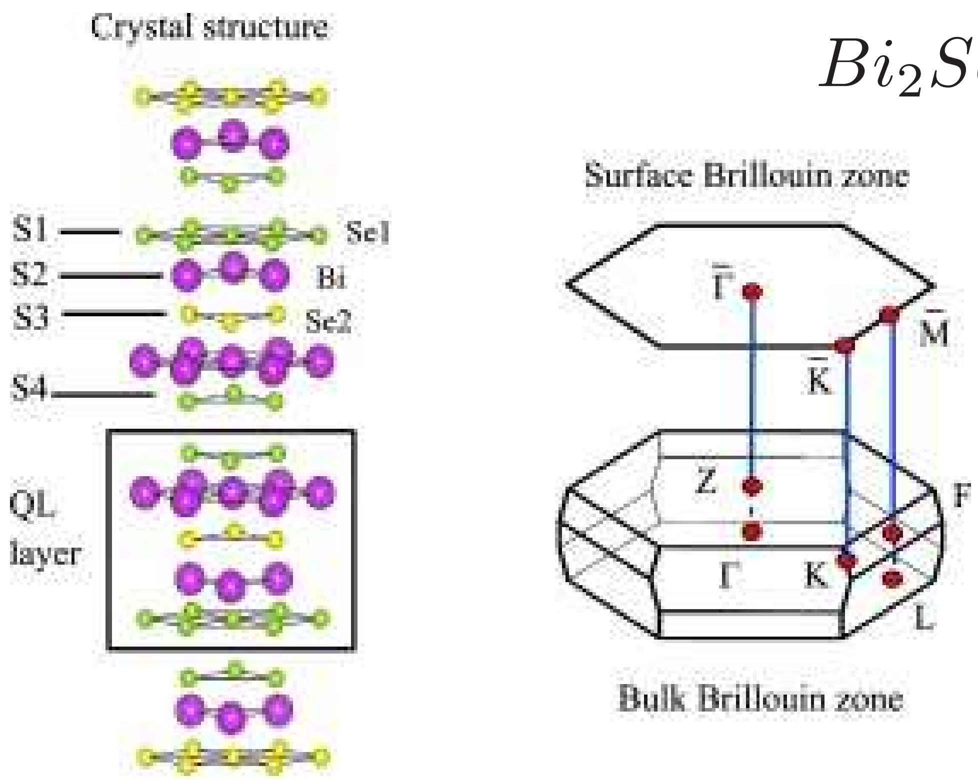
$$m(\boldsymbol{k}) = m_0 + \beta|\boldsymbol{k}|^2$$



Graphene (2D) but represents Staggered fermions

Condensed matter realisation of Dirac fermions on a lattice

Bi_2Se_3



Bismuth selenide as the archetypical example of 3DTI

V. Topological insulators in three dimensions

In 2D (and breaking TRS)

$$J_i = \sigma_H \varepsilon_{ij} E_j$$

Chern Simons term

$$\mathcal{S}_{CS} = \sigma_H \int d^2\boldsymbol{r} dt \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$\sigma_H = \int d^2\boldsymbol{k} \varepsilon_{ij} \partial_i \mathcal{A}_j$$

Berry connection
In momentum space

Dirac 3D is TR symmetric and
Each band is doubly
degenerated due to spin!

$$\mathcal{A}_j^{ab} = i \left\langle u_a \right| \partial_j \left| u_b \right\rangle$$

Non-abelian generalisation of the Berry connection

$$\mathcal{F}_{ij}^{ab} = \partial_i \mathcal{A}_j^{ab} - \partial_j \mathcal{A}_i^{ab} - [\mathcal{A}_j^{ab}, \mathcal{A}_i^{ab}]$$

Non-abelian Berry curvature

V. Topological insulators in three dimensions

In 2D (and breaking TRS)

$$J_i = \sigma_H \varepsilon_{ij} E_j$$

Chern Simons term

$$\mathcal{S}_{CS} = \sigma_H \int d^2\boldsymbol{r} dt \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

Dirac 3D is TR symmetric and
Each band is doubly
degenerated due to spin!

$$\sigma_H = \int d^2\boldsymbol{k} \varepsilon_{ij} \partial_i \mathcal{A}_j$$

Berry connection
In momentum space

Magnetic field

$$\langle P_1 \rangle = \frac{e^2 B_3}{8\pi^3 \hbar} \int_{BZ} d^3\boldsymbol{k} \varepsilon_{ijl} Tr[\mathcal{A}_i \partial_j \mathcal{A}_l - i\frac{2}{3} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_l]$$

Electric polarization

Magnetoelectric effect!

$$\boldsymbol{J} = \frac{d\boldsymbol{P}}{dt}$$

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P} = \boldsymbol{E} + \theta \boldsymbol{B}$$

V. Topological insulators in three dimensions

Magnetoelectric effect!

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} = \boldsymbol{E} + \theta\boldsymbol{B}$$

$$\Delta\mathcal{L} = \theta\boldsymbol{B}\cdot\boldsymbol{E}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho - \boldsymbol{\nabla}\theta\cdot\boldsymbol{B}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \dot{\boldsymbol{E}} + \boldsymbol{J} + \dot{\theta}\boldsymbol{B} + \boldsymbol{\nabla}\theta\times\boldsymbol{E}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\dot{\boldsymbol{B}}$$

Clearly, if θ is constant, there is no physical effect

$$\theta = 0, \pi$$

Normal (→ 0) Topological (→ π)

$$\mathcal{S} = \int d^4x\theta\varepsilon_{ijlr}F_{ij}F_{lr} \propto \int d^4x\theta\partial_i(\varepsilon_{ijlr}A_j\partial_l A_r)$$

Total derivative

V. Topological insulators in three dimensions

$$\mathcal{S} = \int d^4x\theta\varepsilon_{ijlr}F_{ij}F_{lr} \propto \int d^4x\theta\partial_i(\varepsilon_{ijlr}A_j\partial_l A_r)$$

$$\theta = \theta(\boldsymbol{r}, t)$$

$$\mathcal{S} = \int d^4x(\partial_i\theta)\varepsilon_{ijlr}A_j\partial_l A_r$$

$$\partial_i\theta = b_i$$ Three dimensional QAHE

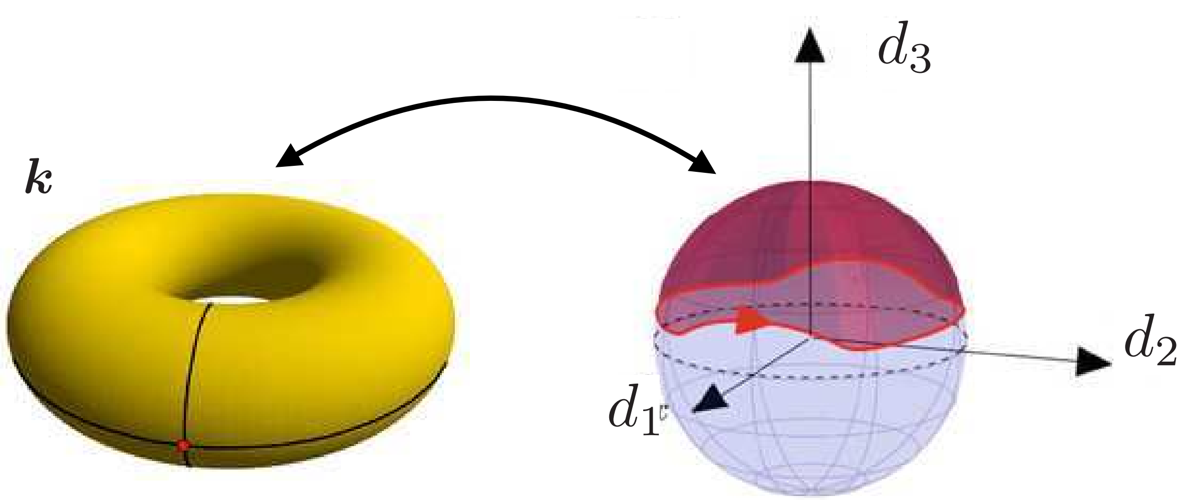
$$\theta = \boldsymbol{b} \cdot \boldsymbol{r}$$ Weyl semimetals

$$\theta = \sigma_H \Theta(z^2 - d^2)$$ Surface QAHE

V. Edge/surface states.

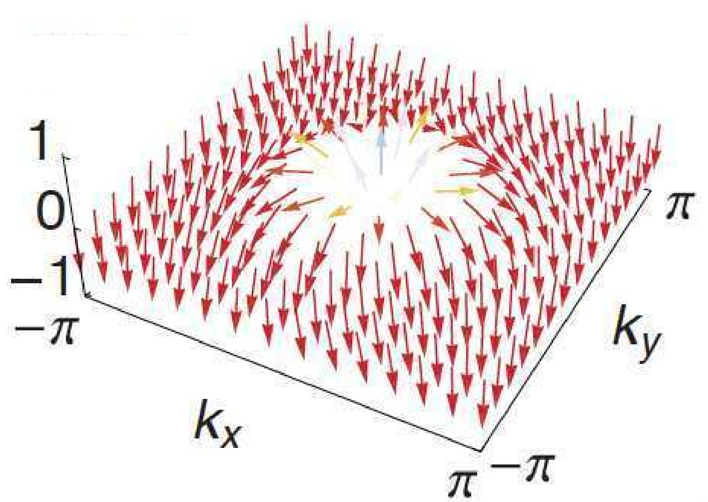
SPT phases need for a symmetry to play with.

They are characterised by a topological invariant defined in some parameter space.

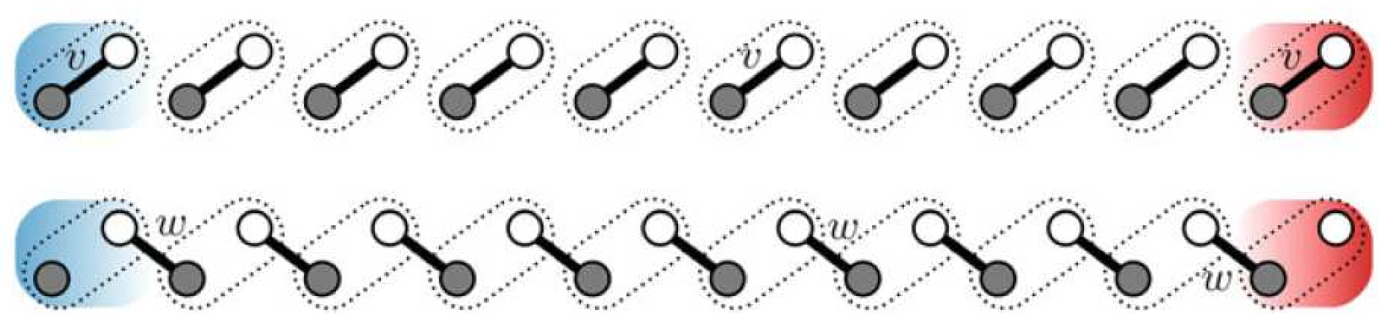


If the case of momentum, if we consider boundaries or defects, translation invariance is lost

Still we can see topology at work: bulk/edge correspondence



$$\nu = 0$$

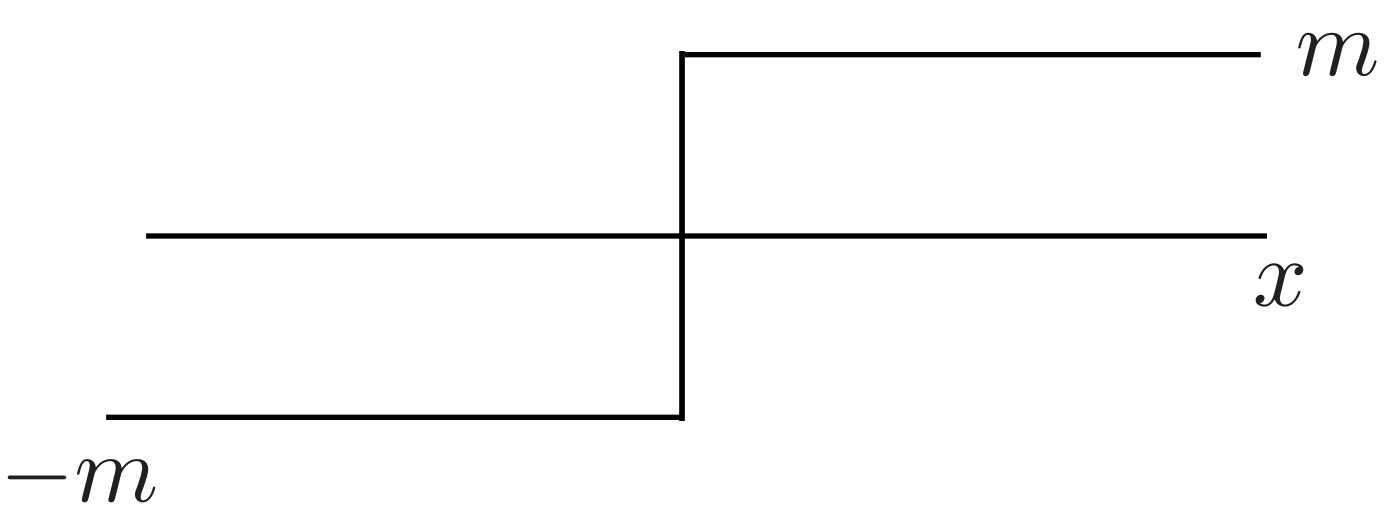


$$\nu \neq 0$$

V. Edge/surface states.

$$\begin{pmatrix} E - v\boldsymbol{\sigma}\cdot\boldsymbol{k} & m(\boldsymbol{k}) \\ m(\boldsymbol{k}) & E + v\boldsymbol{\sigma}\cdot\boldsymbol{k} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = 0$$

$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$



$$H_0(-i\partial_1) = -is\sigma_1\partial_1 + m(x)\sigma_3$$

$$m(x) = \begin{cases} m, x > 0 \\ -m, x < 0 \end{cases}$$

$$\sigma_1\partial_1 \left|\psi\right\rangle = -ism(x)\sigma_3 \left|\psi\right\rangle$$

V. Edge/surface states.

$$\begin{pmatrix} E - v\boldsymbol{\sigma}\cdot\boldsymbol{k} & m(\boldsymbol{k}) \\ m(\boldsymbol{k}) & E + v\boldsymbol{\sigma}\cdot\boldsymbol{k} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = 0$$

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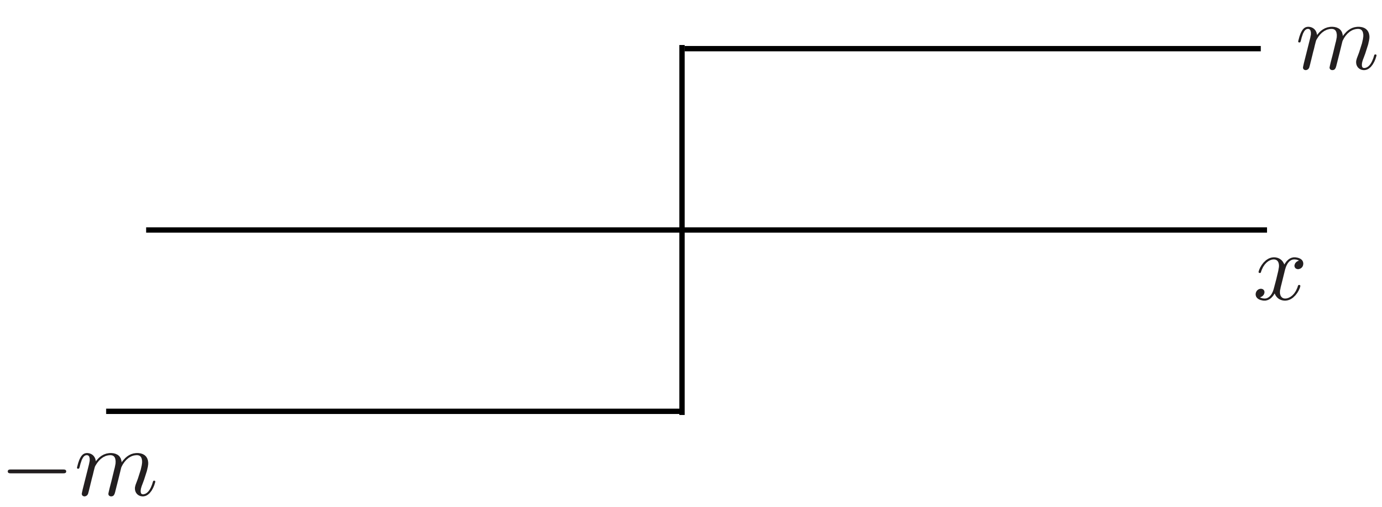

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$$\partial_1 |\psi\rangle = -sm(x)\sigma_2 |\psi\rangle$$

V. Edge/surface states.

$$\begin{pmatrix} E - v\boldsymbol{\sigma}\cdot\boldsymbol{k} & m(\boldsymbol{k}) \\ m(\boldsymbol{k}) & E + v\boldsymbol{\sigma}\cdot\boldsymbol{k} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = 0$$

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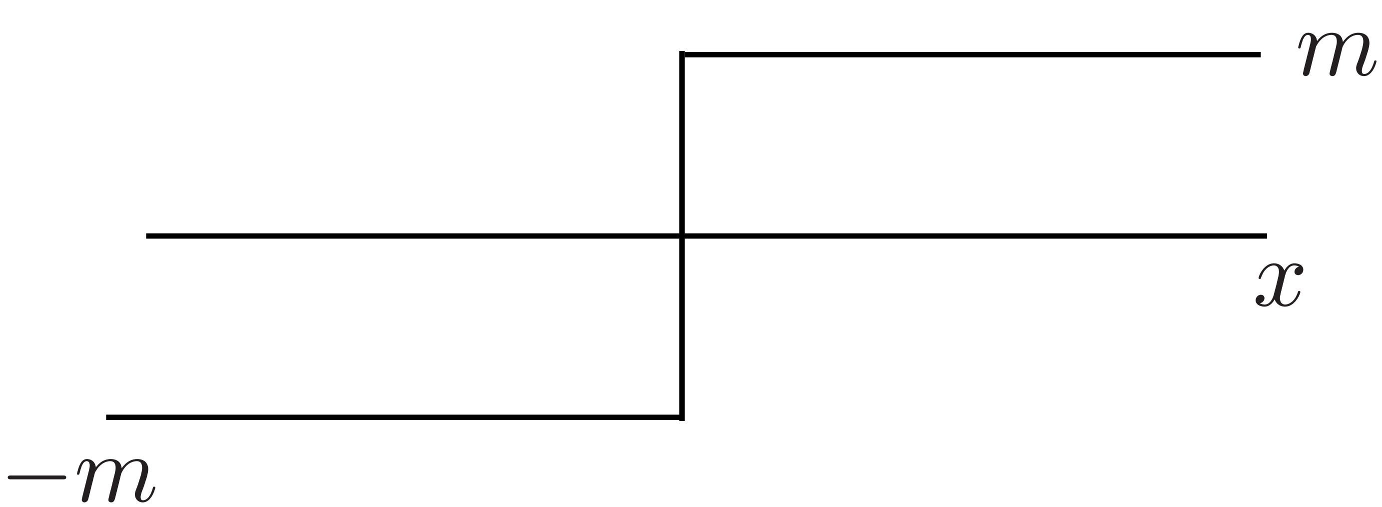


$$H_0(-i\partial_1) = -is\sigma_1\partial_1 + m(x)\sigma_3$$

$$m(x) = \begin{cases} m, x > 0 \\ -m, x < 0 \end{cases}$$

$$\partial_1 |\psi\rangle = -sm(x)\sigma_2 |\psi\rangle$$

$$|\psi\rangle = f(x)\,|s\rangle$$

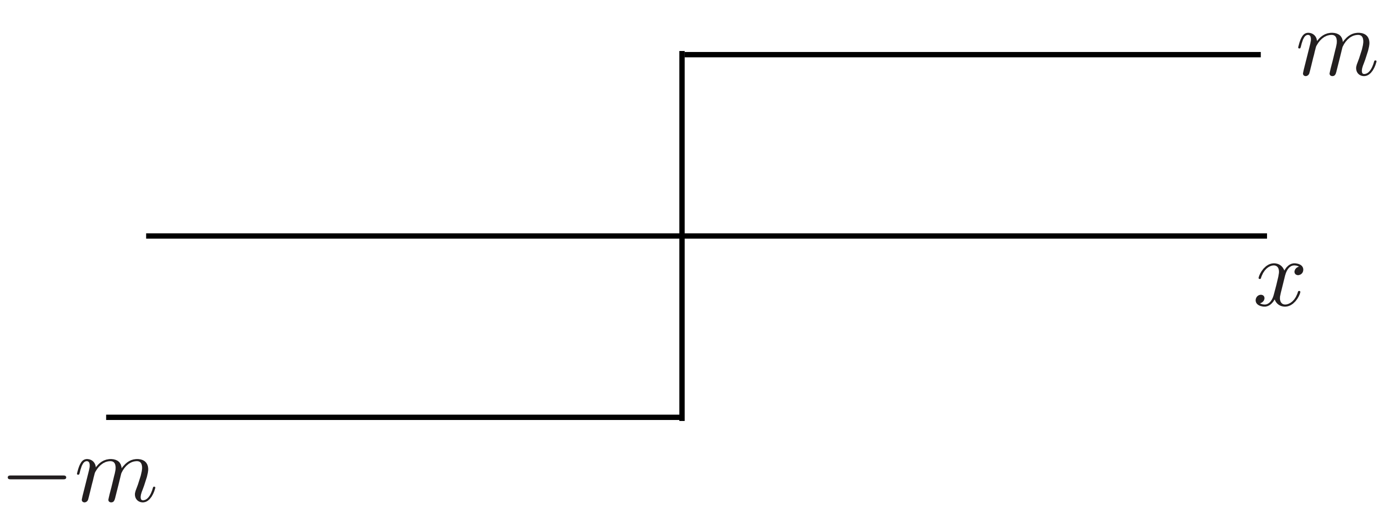
$$\sigma_2 |s\rangle = s\,|s\rangle$$

$$\partial_1 f(x)\,|s'\rangle = -ss'm(x)f(x)\,|s'\rangle$$

V. Edge/surface states.

$$\begin{pmatrix} E - v\boldsymbol{\sigma}\cdot\boldsymbol{k} & m(\boldsymbol{k}) \\ m(\boldsymbol{k}) & E + v\boldsymbol{\sigma}\cdot\boldsymbol{k} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = 0$$

$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$



$$H_0(-i\partial_1) = -is\sigma_1\partial_1 + m(x)\sigma_3$$

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$$\partial_1 |\psi\rangle = -sm(x)\sigma_2 |\psi\rangle$$

$$|\psi\rangle = f(x)|s\rangle$$

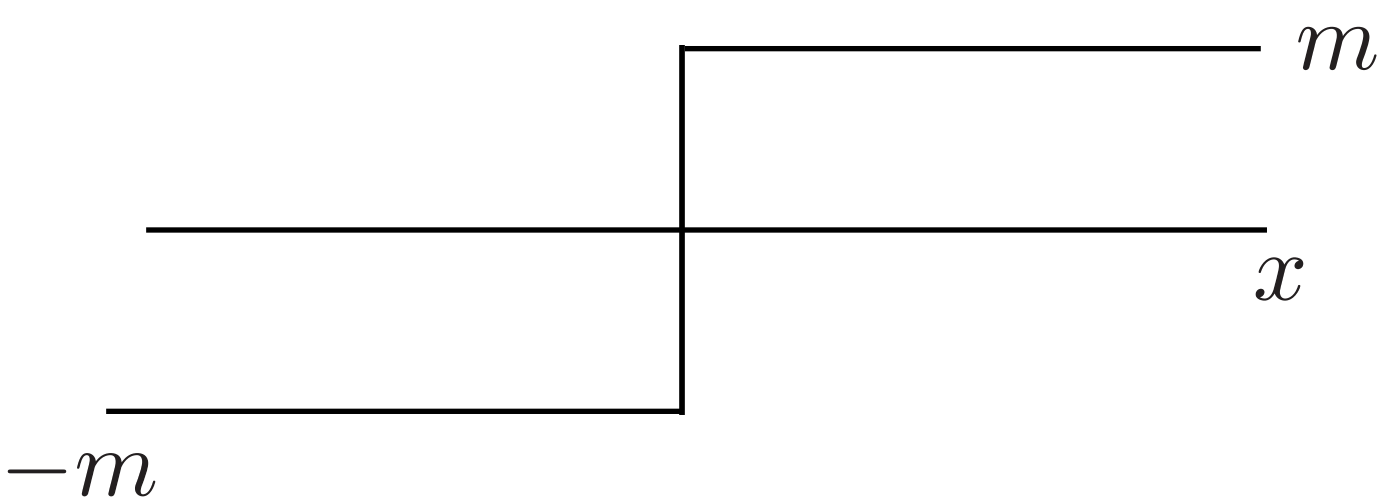
$$f(x) = Ae^{-ss'\int_x dx' m(x')}$$

$$\sigma_2 |s\rangle = s|s\rangle$$

V. Edge/surface states.

$$\begin{pmatrix} E - v\boldsymbol{\sigma}\cdot\boldsymbol{k} & m(\boldsymbol{k}) \\ m(\boldsymbol{k}) & E + v\boldsymbol{\sigma}\cdot\boldsymbol{k} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = 0$$

$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$



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$$|\psi\rangle = f(x)\,|s\rangle$$

$$f(x) = Ae^{-ss|x|}$$

$$\sigma_2 |s\rangle = s\,|s\rangle$$

$$s = s'$$

V. Edge/surface states.

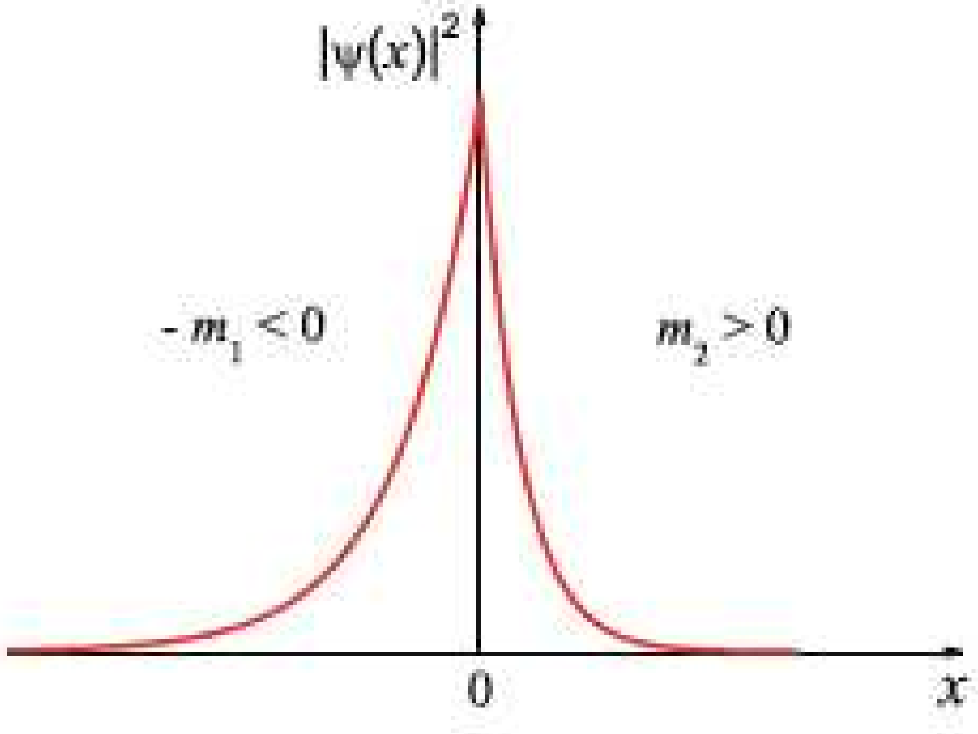
$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$

$$f(x) = Ae^{-ss|x|}$$

$$s = s'$$

$$\sigma_2 \left|s\right\rangle = s \left|s\right\rangle$$

There is a zero energy mode localized at the boundary, with a well defined spin projection



V. Edge/surface states.

$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$

$$f(x) = Ae^{-ss|x|}$$

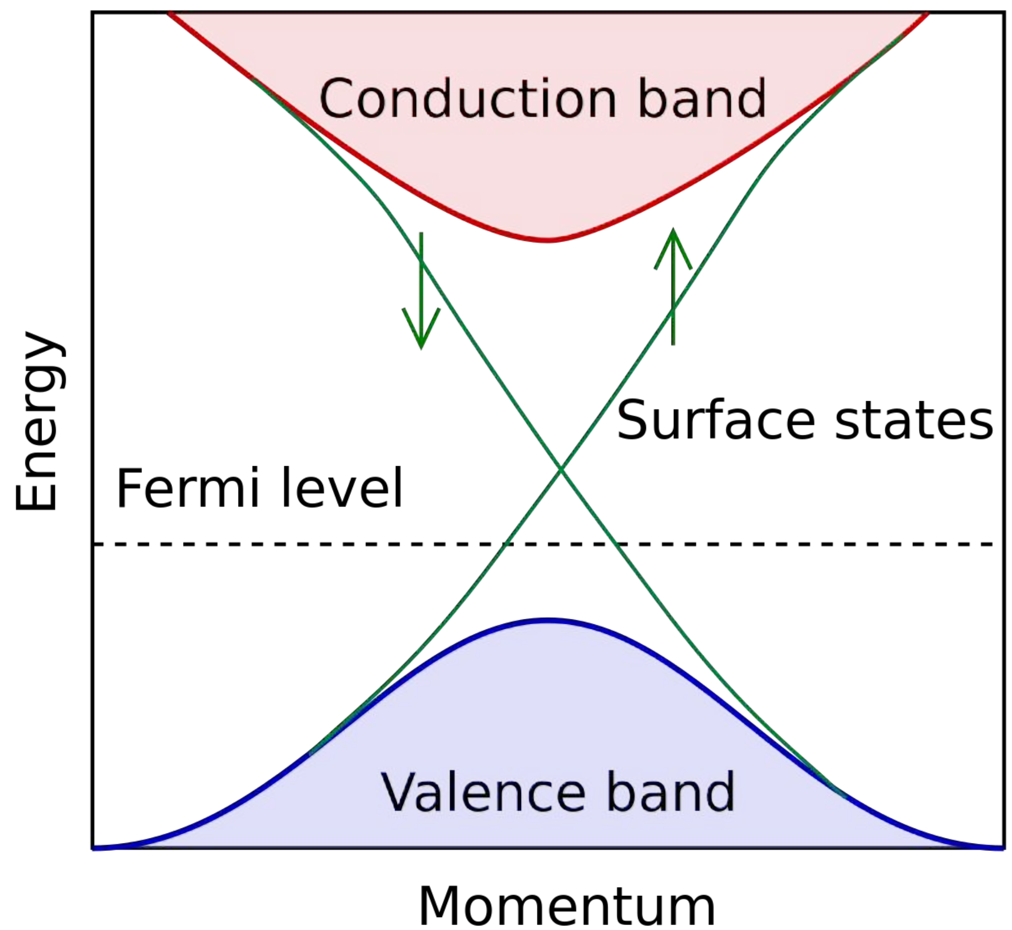
2D:

$$s = s'$$

$$H_{eff}(k_2) = \langle\psi|\, \Delta H(\boldsymbol{k}) \,|\psi\rangle$$

$$\sigma_2 \, |s\rangle = s\, |s\rangle$$

$$\psi = e^{-m|x|} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$



$$H_{eff}(k_2) = k_2\sigma_3$$

Helical edge states

V. Edge/surface states.

$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$

$$f(x) = Ae^{-ss|x|}$$

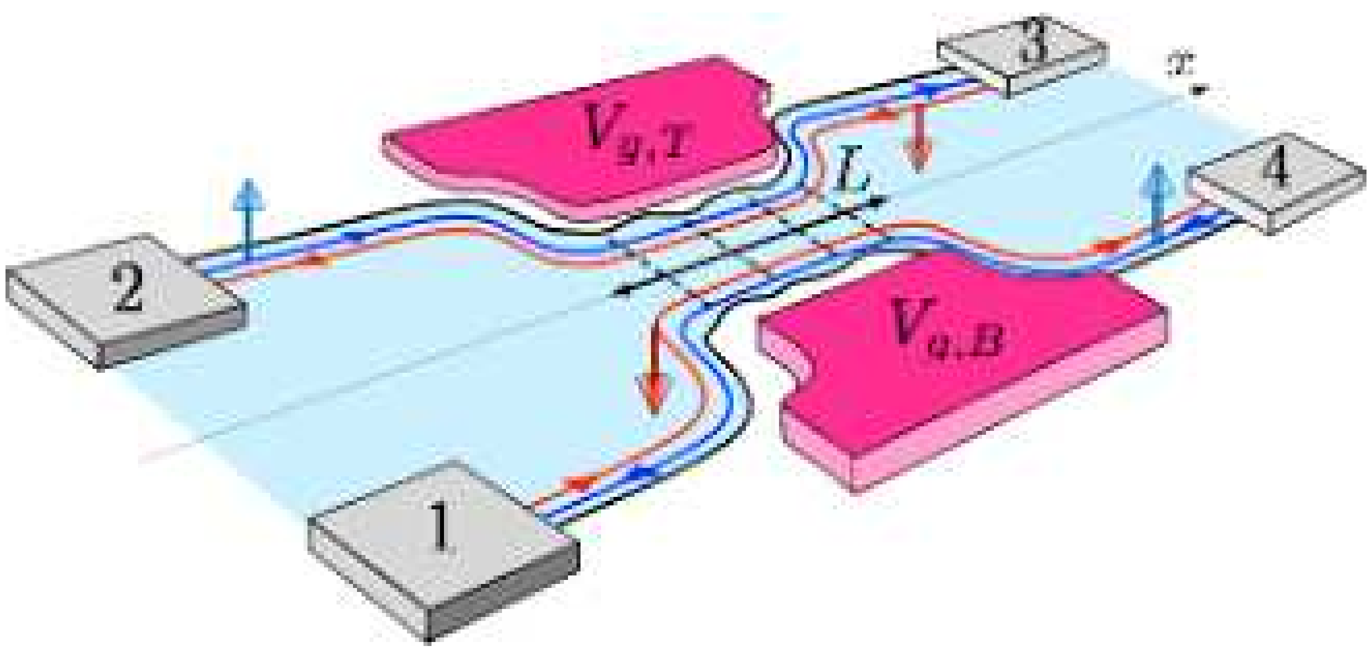
2D:

$$s = s'$$

$$H_{eff}(k_2) = \langle\psi|\,\Delta H(\boldsymbol{k})\,|\psi\rangle$$

$$\sigma_2\,|s\rangle = s\,|s\rangle$$

$$\psi = e^{-m|x|}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$



$$H_{eff}(k_2) = k_2\sigma_3$$

Helical edge states

V. Edge/surface states.

$$H(\boldsymbol{k}) = H_0(-i\partial_1) + \Delta H(\boldsymbol{k})$$

$$f(x) = Ae^{-ss|x|}$$

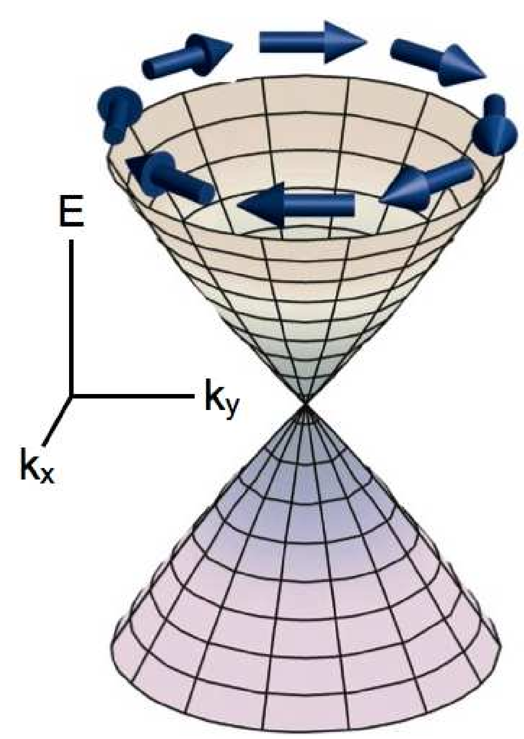
3D:

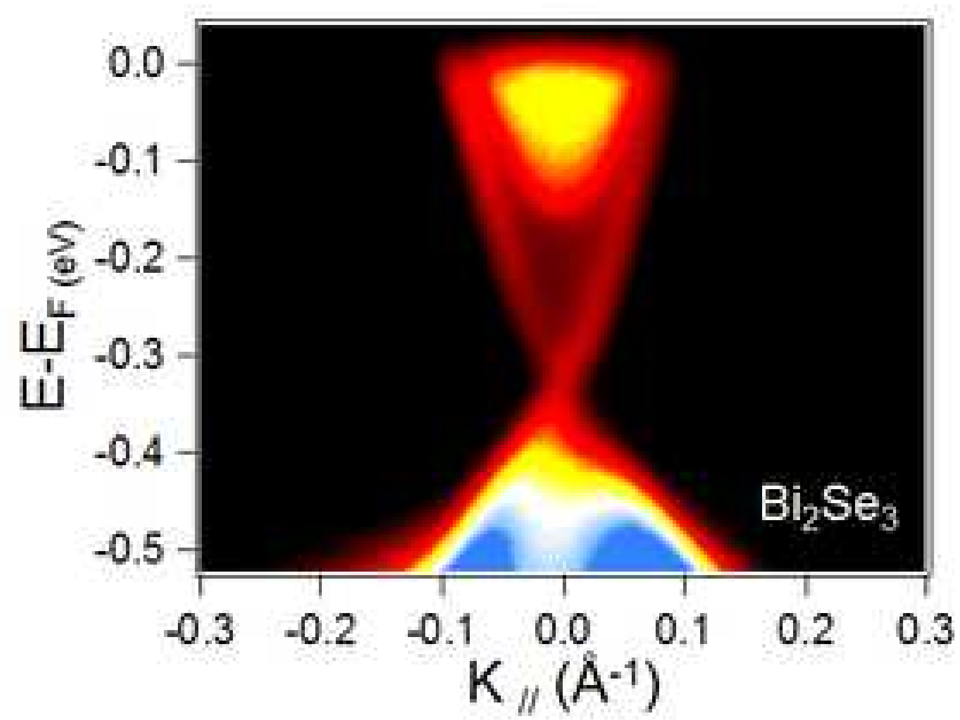
$$s = s'$$

$$H_{eff}(\boldsymbol{k}_\perp) = k_2\sigma_3 - k_3\sigma_2 = \hat{z}\cdot(\boldsymbol{\sigma}\times\boldsymbol{k})$$

$$\sigma_2 |s\rangle = s|s\rangle$$

$$\psi = e^{-m|x|}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$





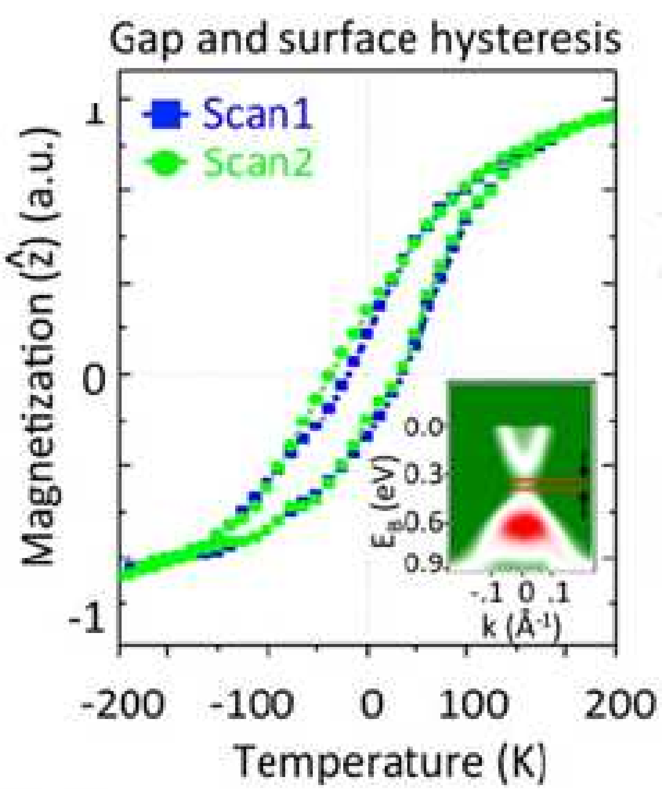
V. Edge/surface states.

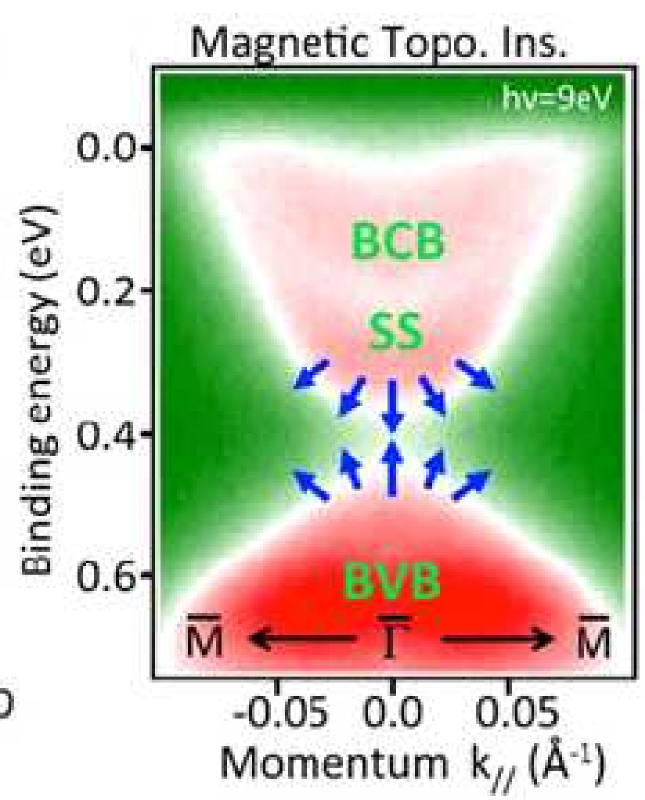
$$H_S(\boldsymbol{k}_\perp) = \hat{z}\cdot(\boldsymbol{\sigma}\times\boldsymbol{k}) + m\sigma_3$$

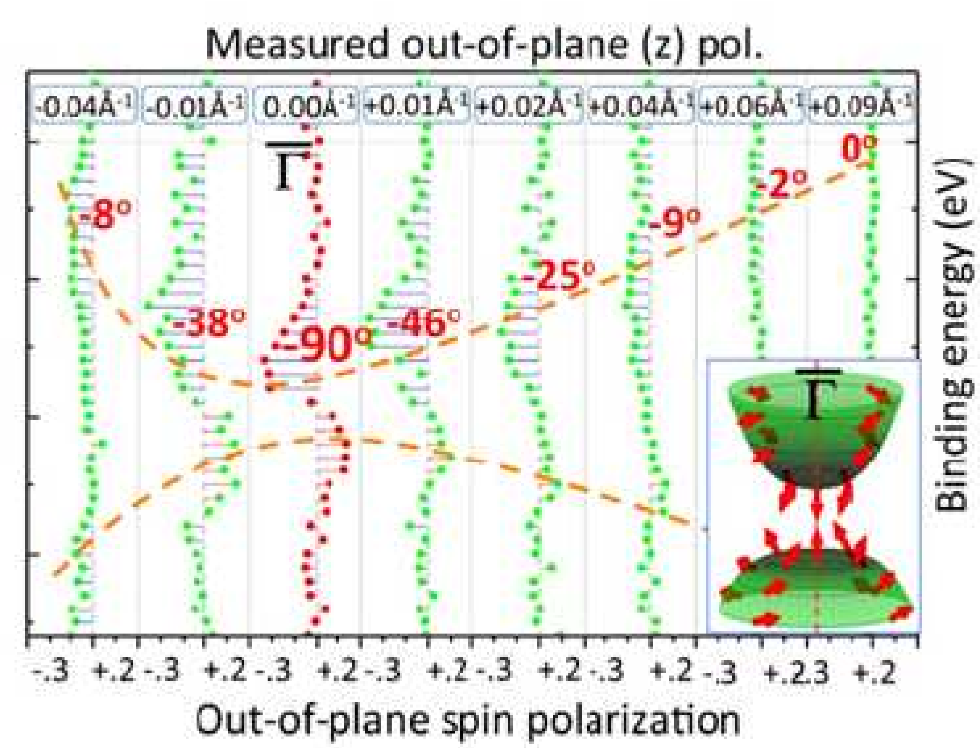
Break TRS to open a gap in the surface state

$$\theta = \sigma_H \Theta(z^2 - d^2)$$

Surface QAHE







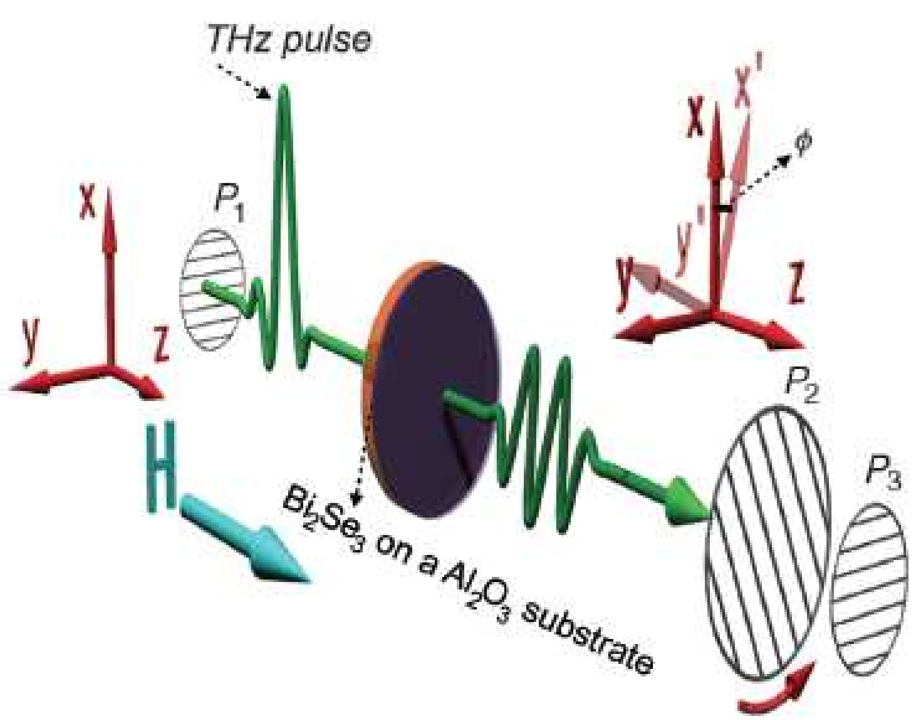
V. Edge/surface states.

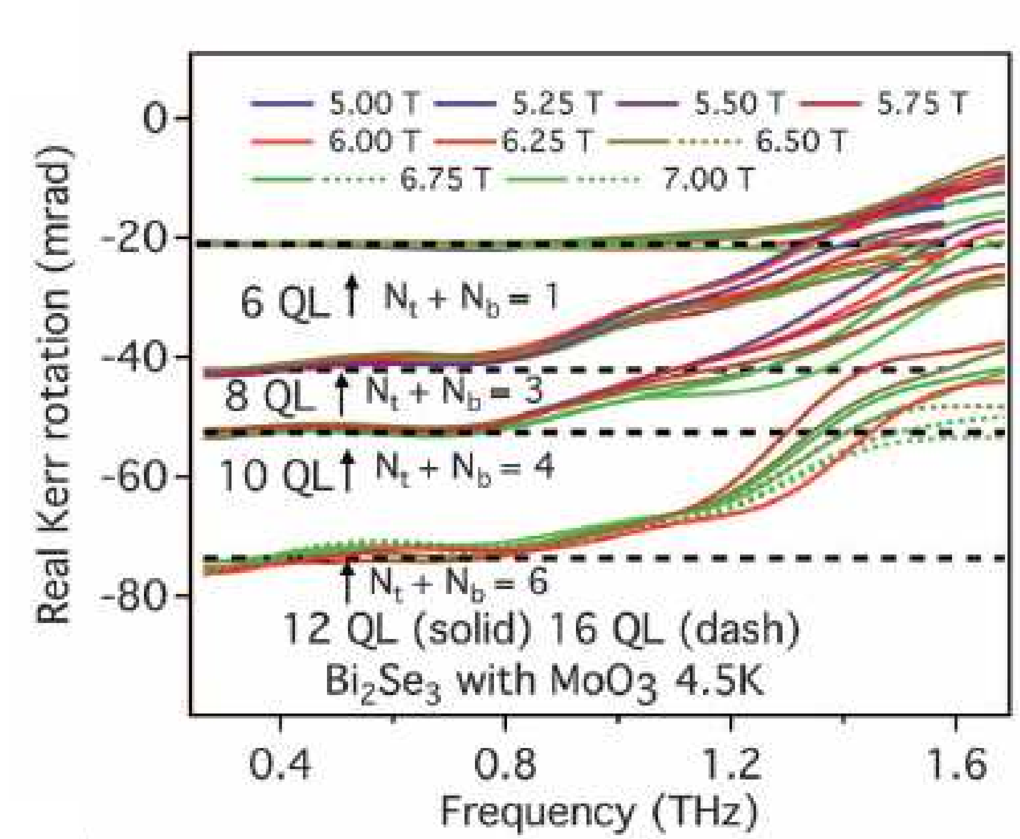
$$H_S(\boldsymbol{k}_\perp) = \hat{z}\cdot(\boldsymbol{\sigma}\times(\boldsymbol{k}-\boldsymbol{A}))$$

Break TRS with B field

$$\theta = \sigma_H \Theta(z^2 - d^2)$$

Surface anomalous QHE (zero LL)





VI. Conclusions to part II

Topological insulators are an example of a type of quantum matter called Symmetry protected topological phases.
What characterizes these phases is the value of a topological invariant and not an order parameter.
In these SPT phases, the topological invariant is a winding number, that, in many cases, can be rephrased in terms of a Berry phase.
Berry phase physics is also compatible with time reversal symmetry and with Higher dimensions.
Most important, the topological invariant "can be measured" (has observational consequences)