Introduction to Topological Superconductivity and Majoranas

Ramón Aguado
Theory of Quantum Materials and Solid State Quantum Technologies
Instituto de Ciencia de Materiales de Madrid ICMM-CSIC
Plataforma de Tecnologías Cuánticas CSIC
ramon.aguado@csic.es

Reviews:

- J. Alicea, Rep. Prog. Phys. 75, 076501(2012).
- Elliot&Franz, Rev. Mod. Phys. 87, 137 (2015).
- R. Aguado, La Rivista del Nuovo Cimento 40, 523 (2017)

Popular article: Ramón Aguado and Leo Kouwenhoven, Physics Today June 2020.

$$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\Psi = 0$$

- Since the γ matrices are in general complex, the solutions of Dirac's equation are complex. This is expected since electrons are electrically charged and this requires complex spinors.

- Majorana's question (1937): does the Dirac equation necessarily involve complex solutions?

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0$$

Majorana's answer: No, if Weyl matrices are purely imaginary (Majorana representation)

$$\gamma^0 = \sigma_y \otimes \sigma_x$$

$$\gamma^1 = i\sigma_x \otimes 1$$

$$\gamma^2 = i\sigma_z \otimes 1$$

$$\gamma^3 = i\sigma_y \otimes \sigma_y$$

$$\Psi = \Psi^*$$

- Neutral particle equals its own antiparticle.
- Very relevant in high-energy physics (neutrinos, dark matter, etc).
- Many experimental efforts to search for Majorana neutrinos are underway. To date their detection remains an experimental challenge.

More rigorously, the particle and antiparticle are charge conjugate solutions of the Dirac equation coupled to the electromagnetic field.

$$\partial_\mu \rightarrow i\partial_\mu + eA_\mu$$
$$\partial_\mu \rightarrow i\partial_\mu - eA_\mu$$

RELEVANT FOR US: in the most general case, it is sufficient to demand in any representation that there exists a matrix such that both charge conjugate solutions are equal

$$\Psi^C = C\Psi^* = \Psi$$

$$C = \tau_y \otimes \sigma_y = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$$

charge conjugation operator

Dirac equation in Weyl representation (chiral spinors where the spin of a particle points in the same direction as its momentum)

$$\begin{pmatrix} -m & i\partial_t - \boldsymbol{\sigma}\mathbf{p} \\ i\partial_t + \boldsymbol{\sigma}\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

$$(i\partial_t - \boldsymbol{\sigma}\mathbf{p})\psi_R - m\psi_L = 0$$

$$(i\partial_t + \boldsymbol{\sigma}\mathbf{p})\psi_L - m\psi_R = 0$$

In the Dirac equation, the mass term mixes left and right chiralities

Dirac equation in Weyl representation (chiral spinors where the spin of a particle points in the same direction as its momentum)

$$\begin{pmatrix} -m & i\partial_t - \boldsymbol{\sigma}\mathbf{p} \\ i\partial_t + \boldsymbol{\sigma}\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

Impose charge conjugation $\Psi^C = \Psi$

$$(i\partial_t - \boldsymbol{\sigma}\mathbf{p})\psi_R - im\sigma_y\psi_R^* = 0$$

$$(i\partial_t + \boldsymbol{\sigma}\mathbf{p})\psi_L - im\sigma_y\psi_L^* = 0$$

In the Majorana equation, chiralities become decoupled and **the mass term mixes particles and antiparticles** (technically, this implies that the equation is not gauge invariant, hence Majorana particles cannot be coupled to the electromagnetic field and are thus necessarily charge neutral).

perspective

NATURE PHYSICS | VOL 5 | SEPTEMBER 2009 | www.nature.com/naturephysics

Majorana returns

Frank Wilczek

www.sciencemag.org SCIENCE VOL 332 8 APRIL 2011

Published by AAAS

Search for Majorana Fermions Nearing Success at Last?

Researchers think they are on the verge of discovering weird new particles that borrow a trick from superconductors and could give a big boost to quantum computers

Physics

Physics **3**, 24 (2010)

Viewpoint

Race for Majorana fermions

Marcel Franz

Department of Physics and Astronomy, University of British Columbia, Va

Published March 15, 2010

The race for realizing Majorana fermions elusive particles th
we still await ideal materials to work with.

Physics

Physics **4**, 67 (2011)

Viewpoint

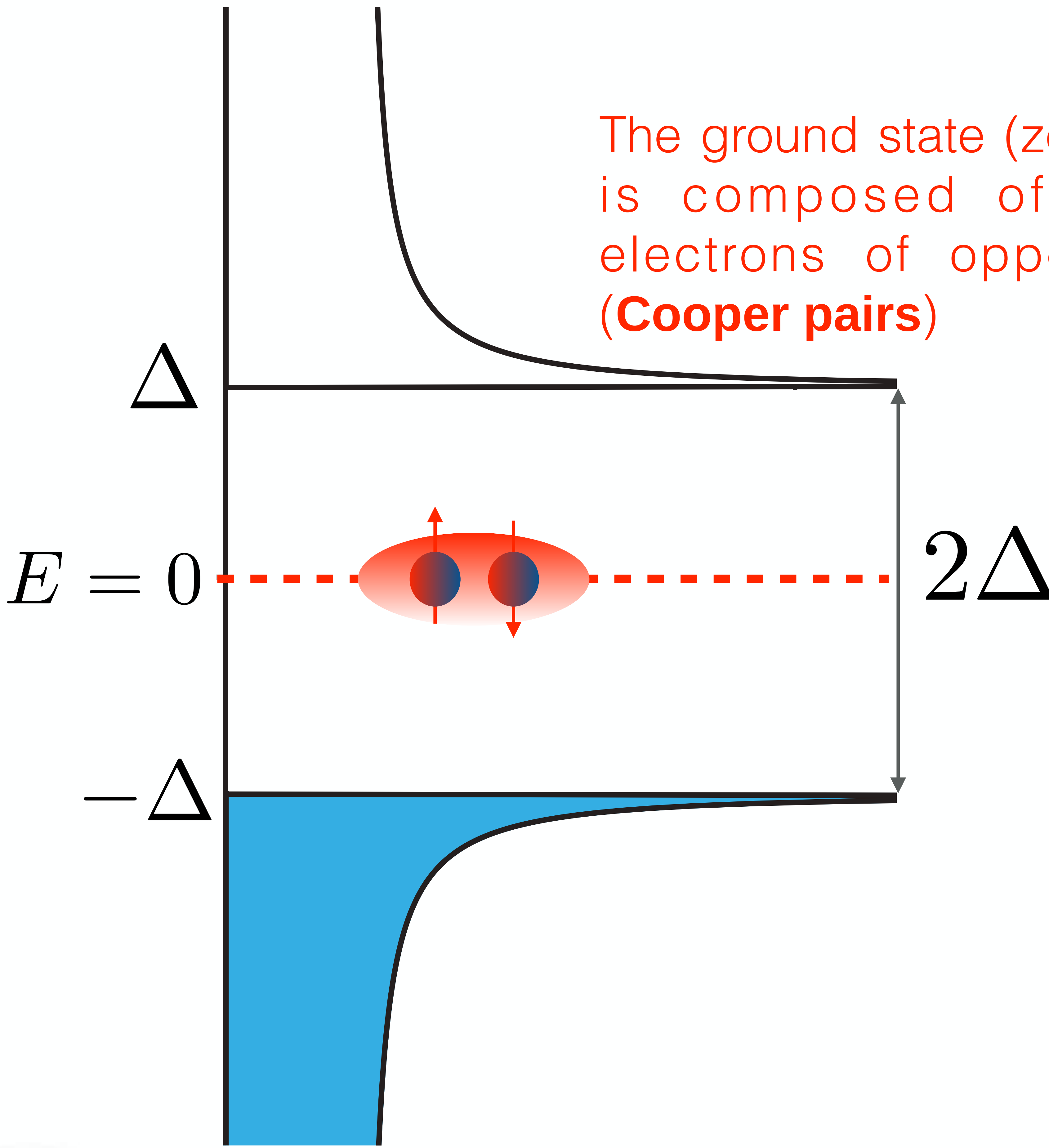
Majorana fermions inch closer to reality

Taylor L. Hughes

University of Illinois at Urbana-Champaign, 1110 W. Green St., Urbana, IL 61801, USA

Published August 22, 2011

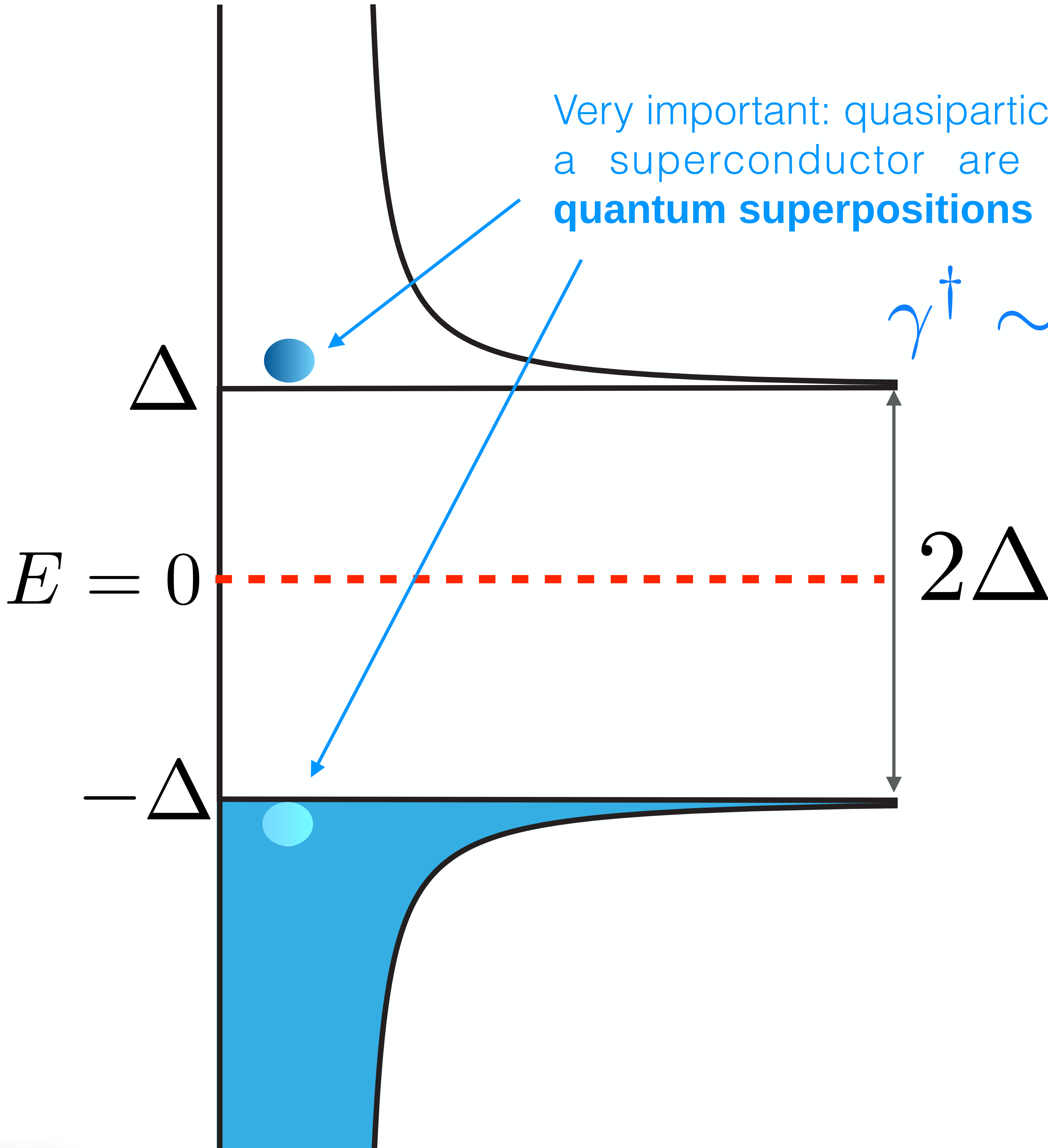
ALL PROPOSALS ARE BASED ON SYSTEMS INVOLVING **SUPERCONDUCTORS, WHY?**

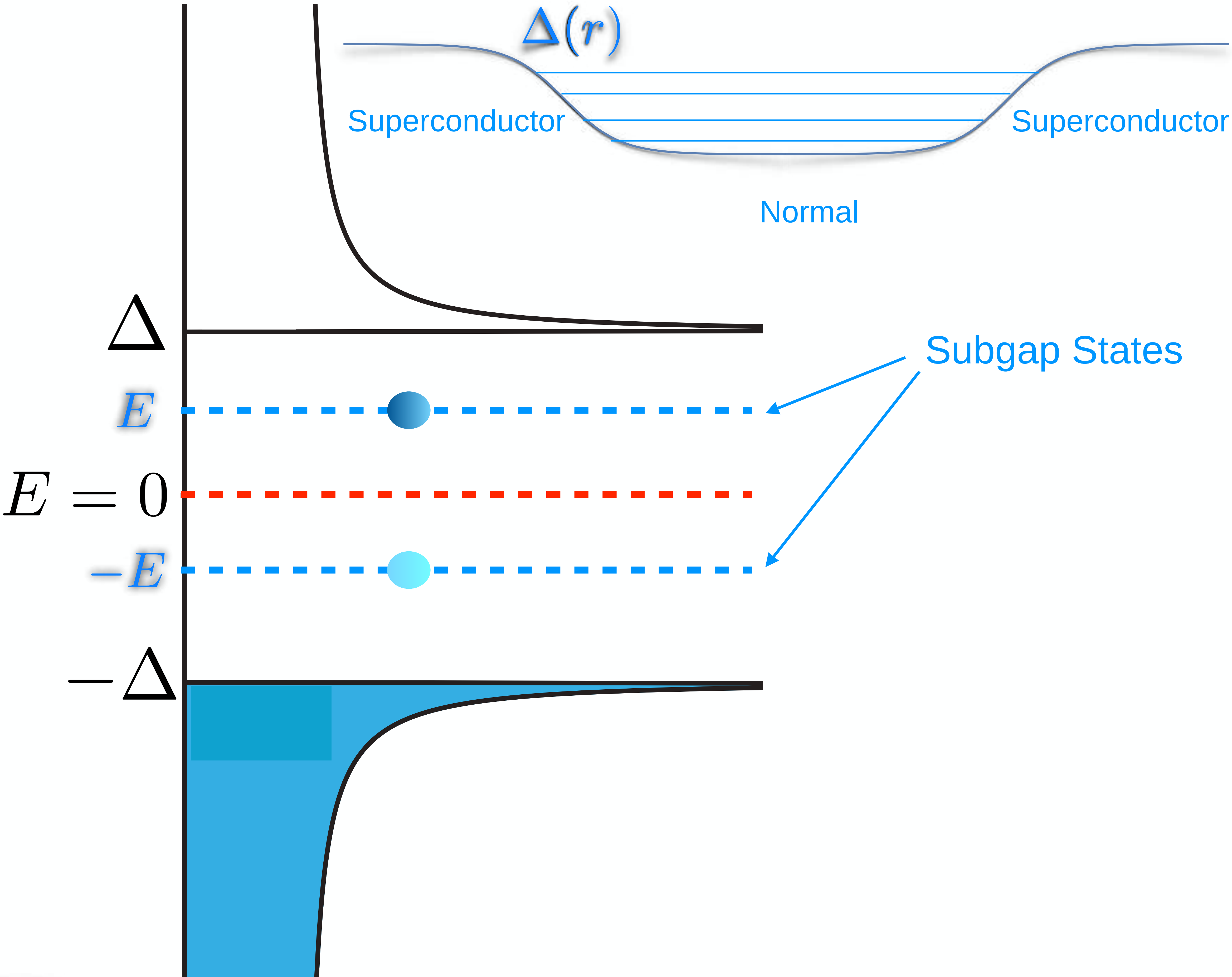


The ground state (zero energy) is composed of pairs of electrons of opposite spin (**Cooper pairs**)

Very important: quasiparticle excitations in a superconductor are **electron-hole quantum superpositions**

$$\gamma^\dagger \sim u c^\dagger + v c$$





Ramón Aguado ICMM-CSIC

Superconducting BCS mean-field pairing for a generic Hamiltonian

$$\mathcal{H} = \int d\mathbf{r}[\sum_{\sigma,\sigma'} H_0^{\sigma,\sigma'}(\mathbf{r})c^\dagger_{\sigma\mathbf{r}}c_{\sigma'\mathbf{r}} + (\Delta(\mathbf{r})c^\dagger_{\uparrow\mathbf{r}}c^\dagger_{\downarrow\mathbf{r}} + H.c.)].$$

Four-component Nambu spinor
$$\hat{\Psi}(\mathbf{r}) = \begin{pmatrix} c_{\uparrow,\mathbf{r}} \\ c_{\downarrow,\mathbf{r}} \\ -c^\dagger_{\downarrow,\mathbf{r}} \\ c^\dagger_{\uparrow,\mathbf{r}} \end{pmatrix}$$

Bogoliubov-DeGennes (BdG) hamiltonian, which is a **redundant** description to treat electrons and holes at the same footing

$$\mathcal{H} = \frac{1}{2}\int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})H_{BdG}(\mathbf{r})\hat{\Psi}(\mathbf{r}),$$

Bogoliubov-DeGennes (BdG) hamiltonian, which is a **redundant** description to treat electrons and holes at the same footing

$$\mathcal{H} = \frac{1}{2}\int d\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r})H_{BdG}(\mathbf{r})\hat{\Psi}(\mathbf{r}),$$

Electron sector

$$H_{BdG}(\mathbf{r}) = \begin{pmatrix} H_0(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -\sigma^y H_0^*(\mathbf{r})\sigma^y \end{pmatrix}$$

hole sector (time-reversed version of the electron)

The redundancy is resolved by observing that the BdG Hamiltonian has **particle-hole (charge conjugation) symmetry**.

$$C = \tau^y \sigma^y = \begin{pmatrix} 0 & -i\sigma^y \\ i\sigma^y & 0 \end{pmatrix}$$

Charge conjugation operator

$$C H^*_{BdG}(r) C^\dagger = -H_{BdG}(r)$$

For each eigenstate at positive energy there is an **equivalent particle-hole conjugate eigenstate at negative energy**

Bogoliubov-DeGennes (BdG) hamiltonian, which is a **redundant** description to treat electrons and holes at the same footing

$$H_{BdG}(\mathbf{r})\Phi_n(\mathbf{r}) = E\Phi_n(\mathbf{r}),$$

$$\Phi_n(\mathbf{r}) = [u_{n\uparrow}(\mathbf{r}), u_{n\downarrow}(\mathbf{r}), v_{n\uparrow}(\mathbf{r}), v_{n\downarrow}(\mathbf{r})]^T$$

Stationary solutions

$$\mathcal{H} = \frac{1}{2}\sum_n E_n \gamma_n^\dagger \gamma_n$$

$$\gamma_n = \int d\mathbf{r}\Phi_n^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}) = \int d\mathbf{r}[u_{n,\uparrow}^*(\mathbf{r})c_{\mathbf{r},\uparrow} + u_{n,\downarrow}^*(\mathbf{r})c_{\mathbf{r},\downarrow} - v_{n,\uparrow}^*(\mathbf{r})c_{\mathbf{r},\downarrow}^\dagger + v_{n,\downarrow}^*(\mathbf{r})c_{\mathbf{r},\uparrow}^\dagger]$$

$$\mathcal{H} = \frac{1}{2}\sum_n E_n \gamma_n^\dagger \gamma_n$$

$$\gamma_n = \int d\mathbf{r}\Phi_n^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}) = \int d\mathbf{r}[u_{n,\uparrow}^*(\mathbf{r})c_{\mathbf{r},\uparrow} + u_{n,\downarrow}^*(\mathbf{r})c_{\mathbf{r},\downarrow} - v_{n,\uparrow}^*(\mathbf{r})c_{\mathbf{r},\downarrow}^\dagger + v_{n,\downarrow}^*(\mathbf{r})c_{\mathbf{r},\uparrow}^\dagger]$$

Redundancy in the BdG Hamiltonian

Same excitation

Positive energy	Negative energy
$\gamma_n^\dagger$	$\gamma_m^\dagger = \gamma_n$
Positive energy	Negative energy
E_n	$E_m = -E_n$

$$\Phi_{E_m}(\mathbf{r}) = C\Phi_{E_n}^*(\mathbf{r})$$

$$\Phi_{E_m}(\mathbf{r}) = C\Phi^*_{E_n}(\mathbf{r})$$
$$\gamma^\dagger_m = \gamma_n$$
$$E_m = -E_n = E$$

- These symmetries originate from the Nambu spinor whose form is in turn dictated by the structure of the second quantized BdG Hamiltonian. **BdG theory has Majorana character built in!**

$$\hat{\Psi}(\mathbf{r}) = C\hat{\Psi}^*(\mathbf{r}) = \begin{pmatrix} \hat{\psi}(\mathbf{r}) \\ i\sigma^y\hat{\psi}^*(\mathbf{r}) \end{pmatrix}$$

  - Note however that this Majorana property is fulfilled by the entire field operator **<u>not</u>** by the eigenmodes of well defined energy (<u>which is what we typically probe in an experiment</u>).

Standard BdG eingenmode

$$\Phi_{E_m}(\mathbf{r}) = C\Phi^*_{E_n}(\mathbf{r})$$

$$\gamma_m^\dagger = \gamma_n$$

$$E_m = -E_n = E$$

zero mode

$$\gamma_m^\dagger = \gamma_m$$

$$E_m = -E_m = 0$$

$$\Phi_0(\mathbf{r}) = C\Phi^*_0(\mathbf{r})$$

$$\Phi_0(\mathbf{r}) = C\Phi_0^*(\mathbf{r})$$

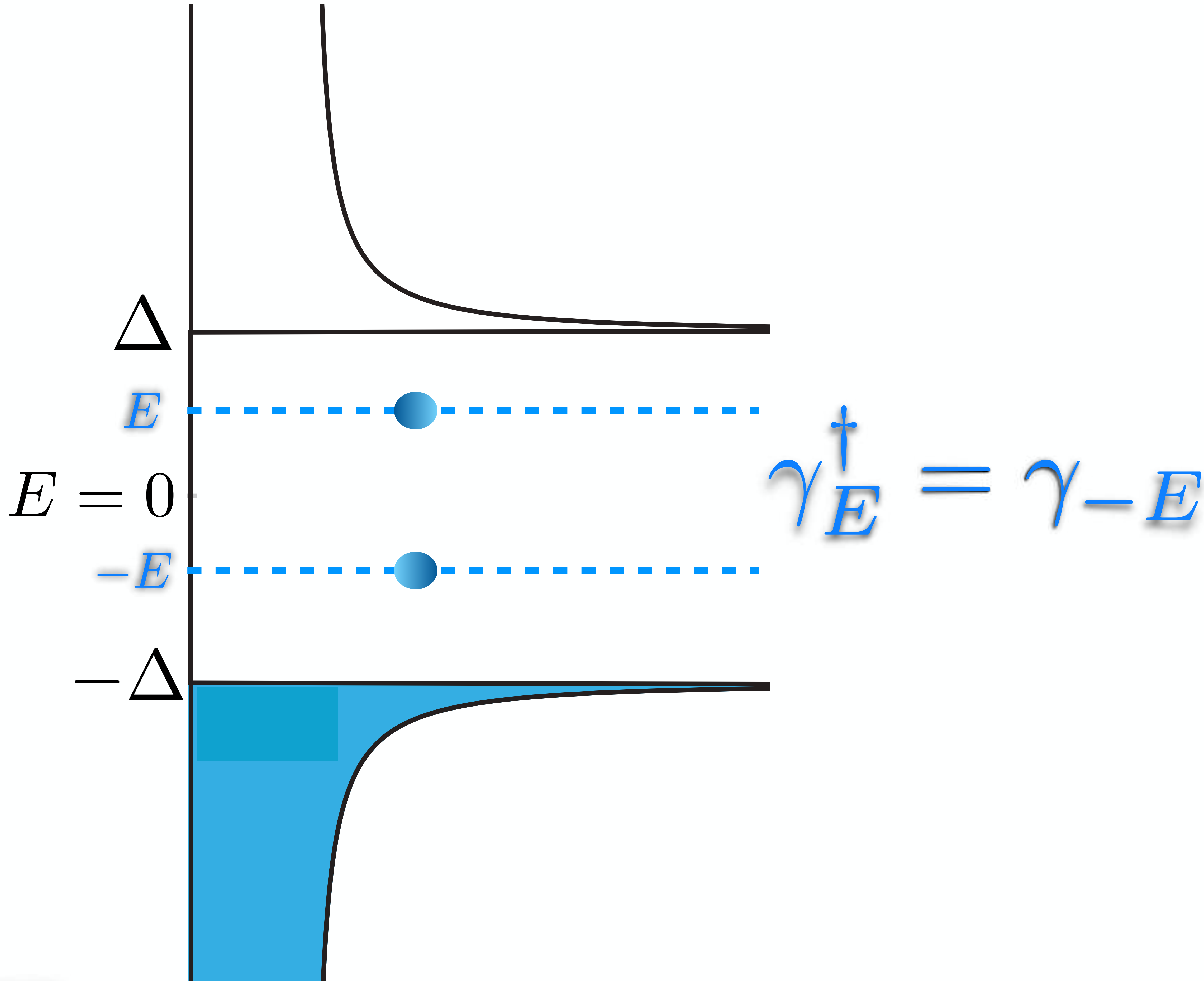
$$\gamma_0^\dagger = \gamma_0$$

$$E_0 = 0$$

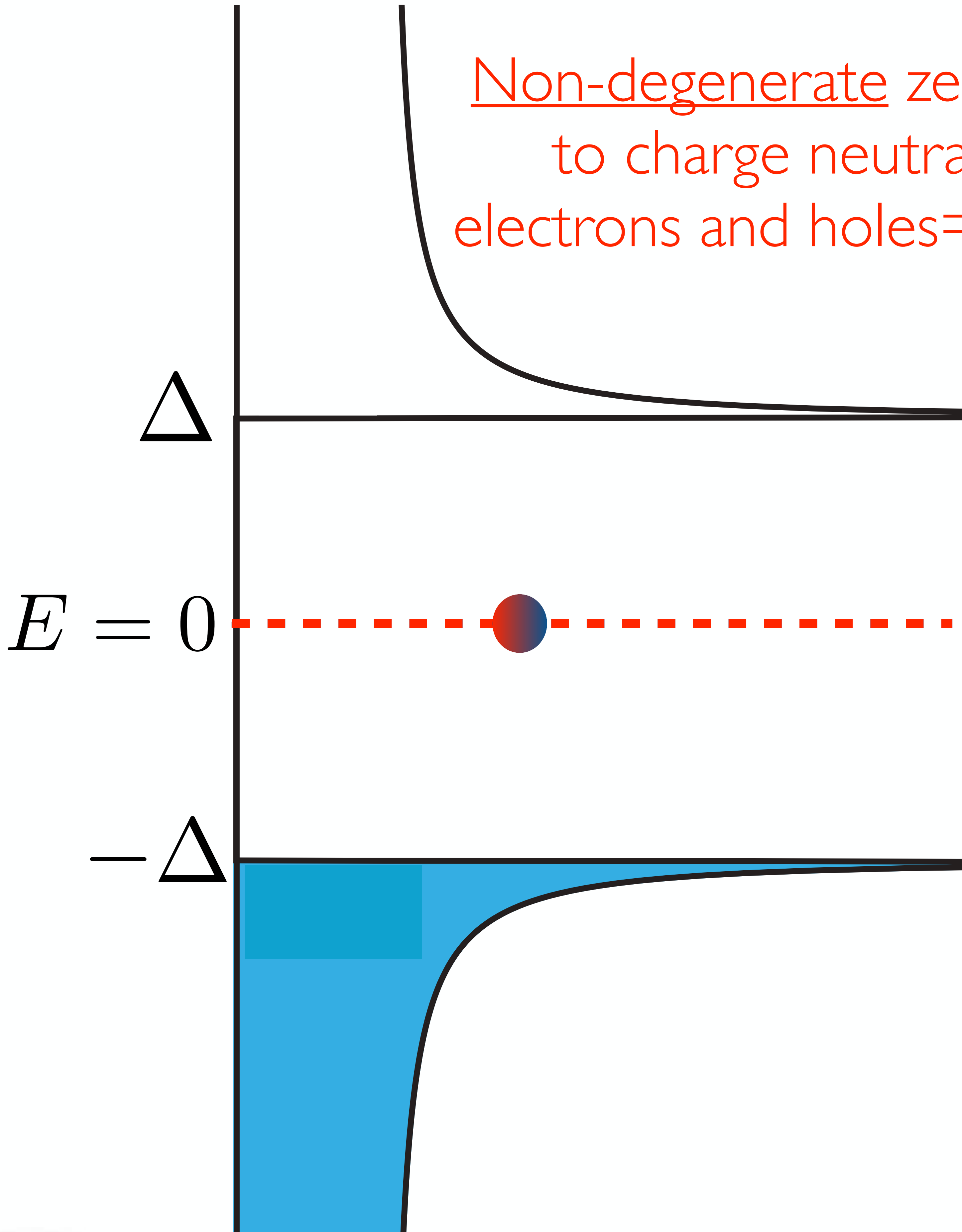
$$\gamma_0 = i\int d\mathbf{r}[u_{0,\uparrow}^*(\mathbf{r})c_{\mathbf{r},\uparrow} + u_{0,\downarrow}^*(\mathbf{r})c_{\mathbf{r},\downarrow} - u_{0,\downarrow}(\mathbf{r})c_{\mathbf{r},\downarrow}^\dagger - u_{0,\uparrow}(\mathbf{r})c_{\mathbf{r},\uparrow}^\dagger]$$

Which is clearly self-conjugate

Majorana Zero Modes

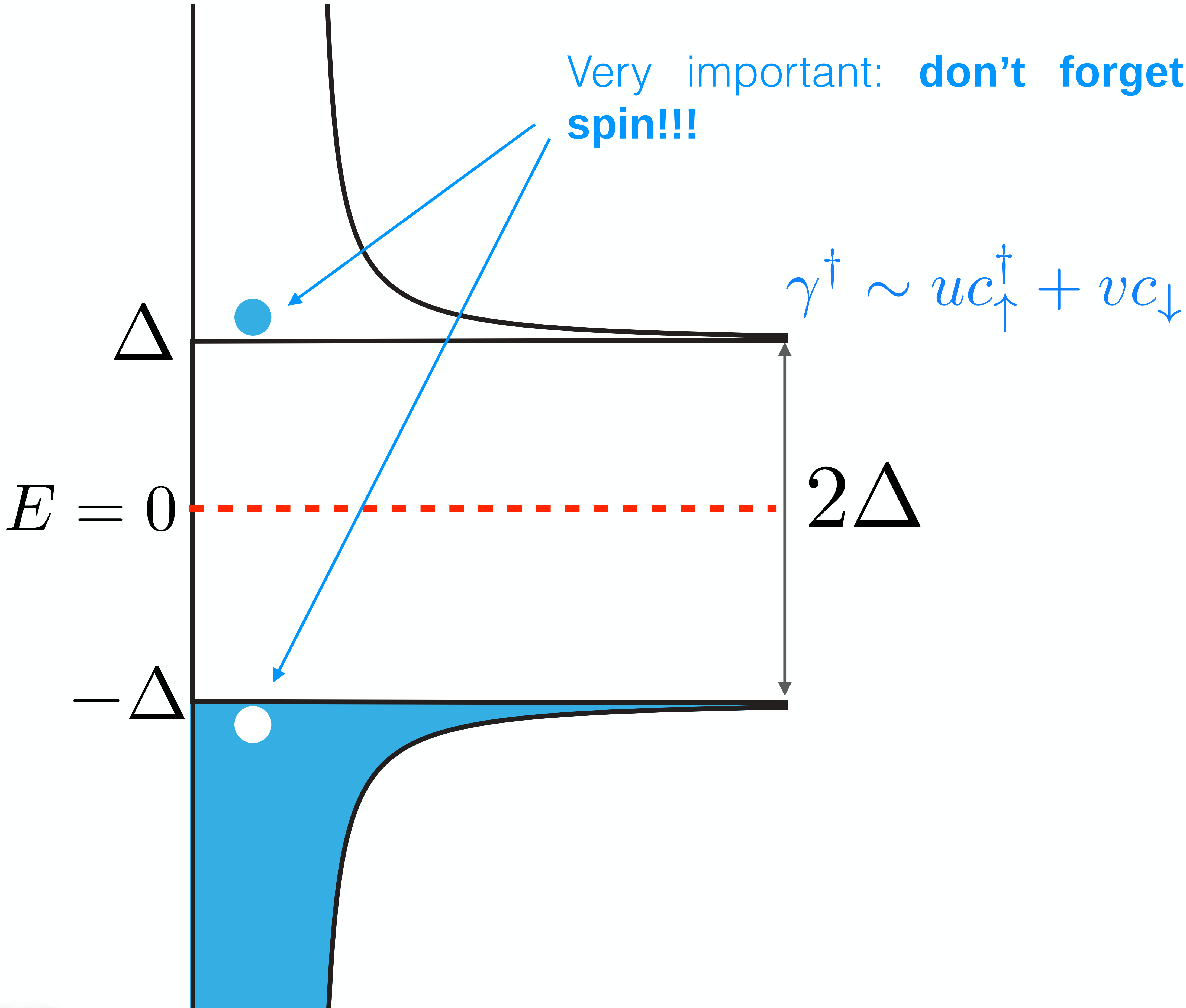


Majorana Zero Modes



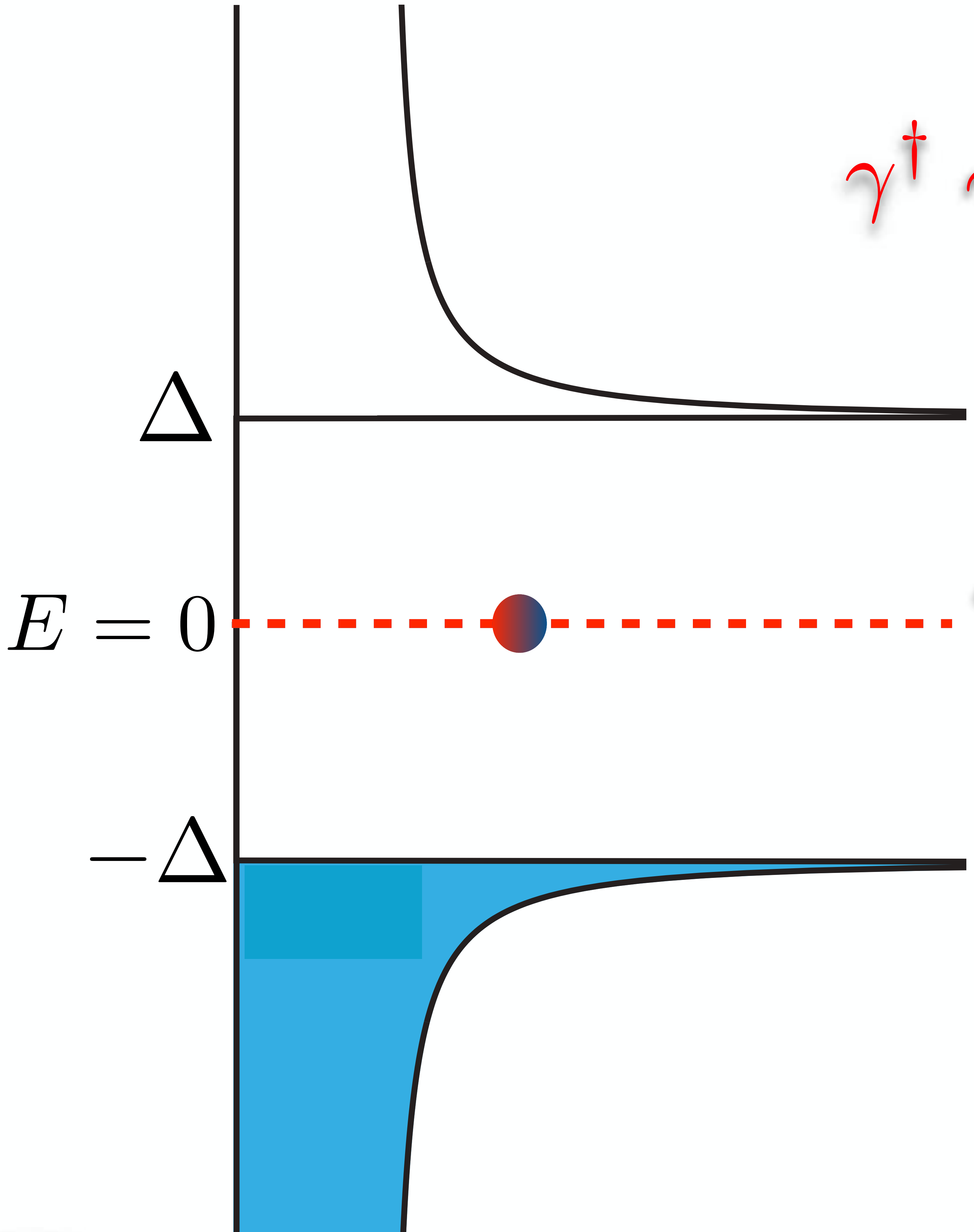
Non-degenerate zero modes correspond to charge neutral superpositions of electrons and holes=Majorana zero modes

$$\gamma_0^\dagger = \gamma_0$$



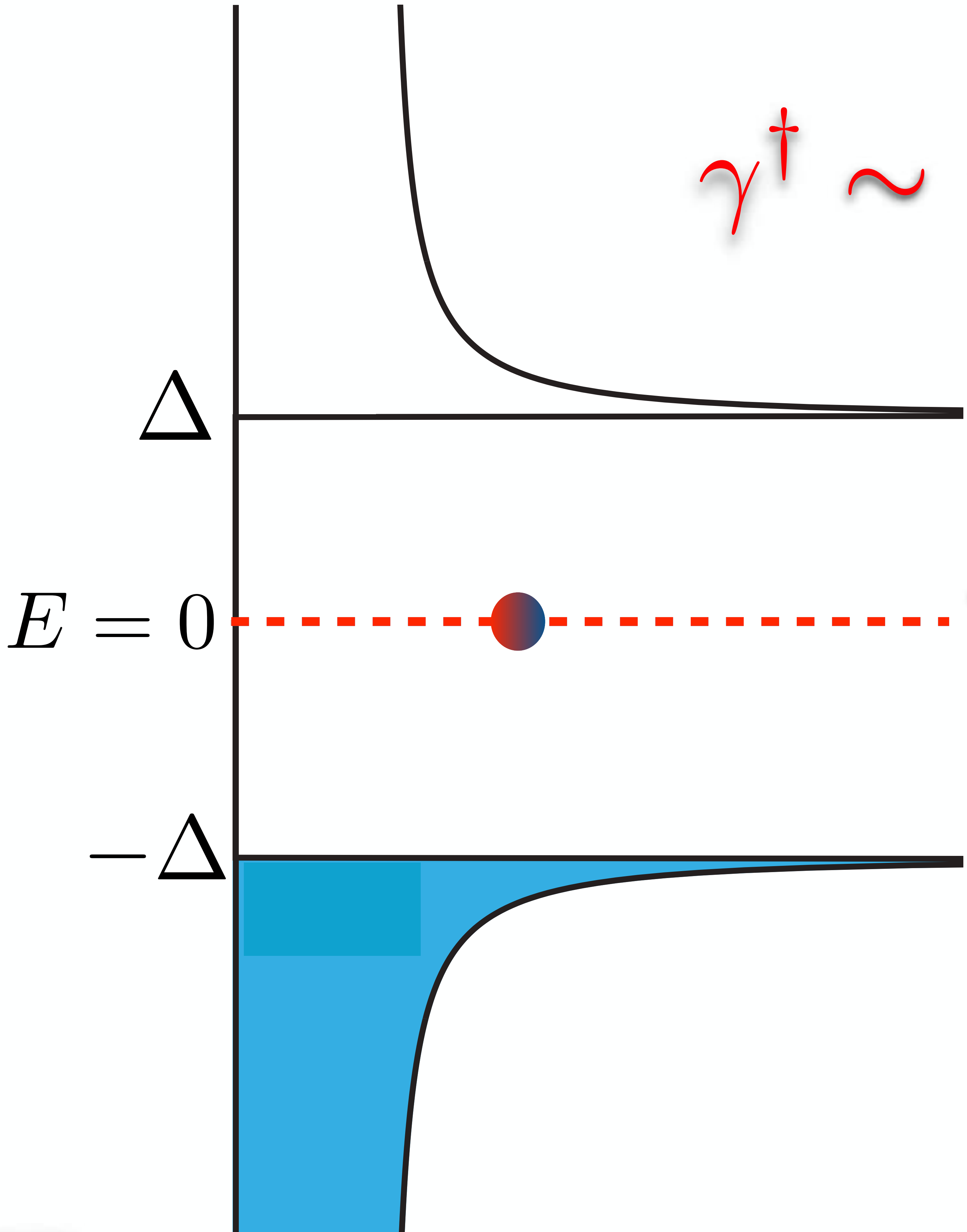
Very important: **don't forget spin!!!**
$\gamma^\dagger \sim u c_\uparrow^\dagger + v c_\downarrow$
Δ
$E = 0$
$-\Delta$
2Δ

Majorana Zero Modes



$$\gamma^{\dagger} \sim \frac{1}{2}(c_{\uparrow}^{\dagger} + c_{\downarrow})$$

$$\gamma_0^{\dagger} \neq \gamma_0$$

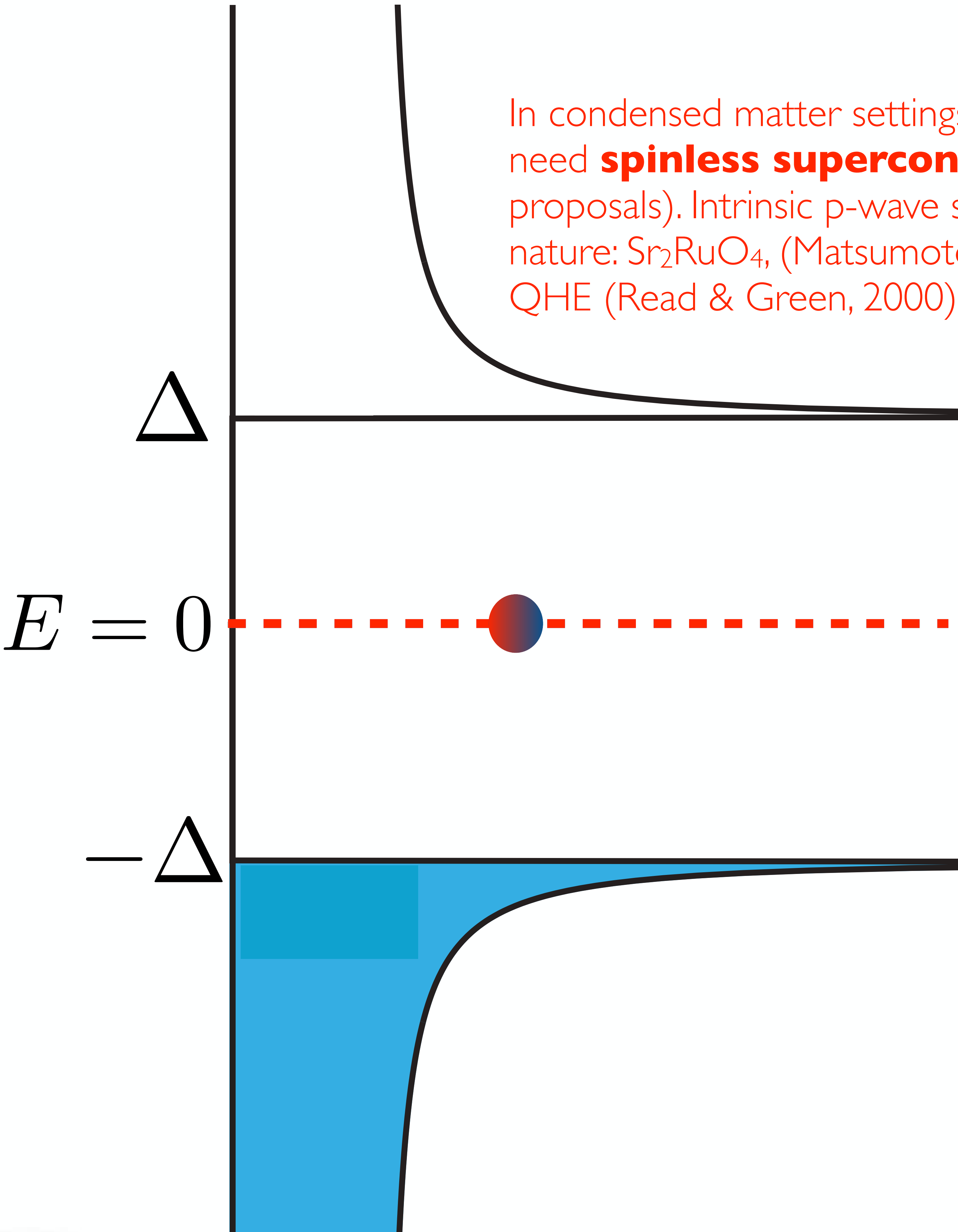


$$\gamma^\dagger \sim \frac{1}{2}(c_\uparrow^\dagger + c_\uparrow)$$

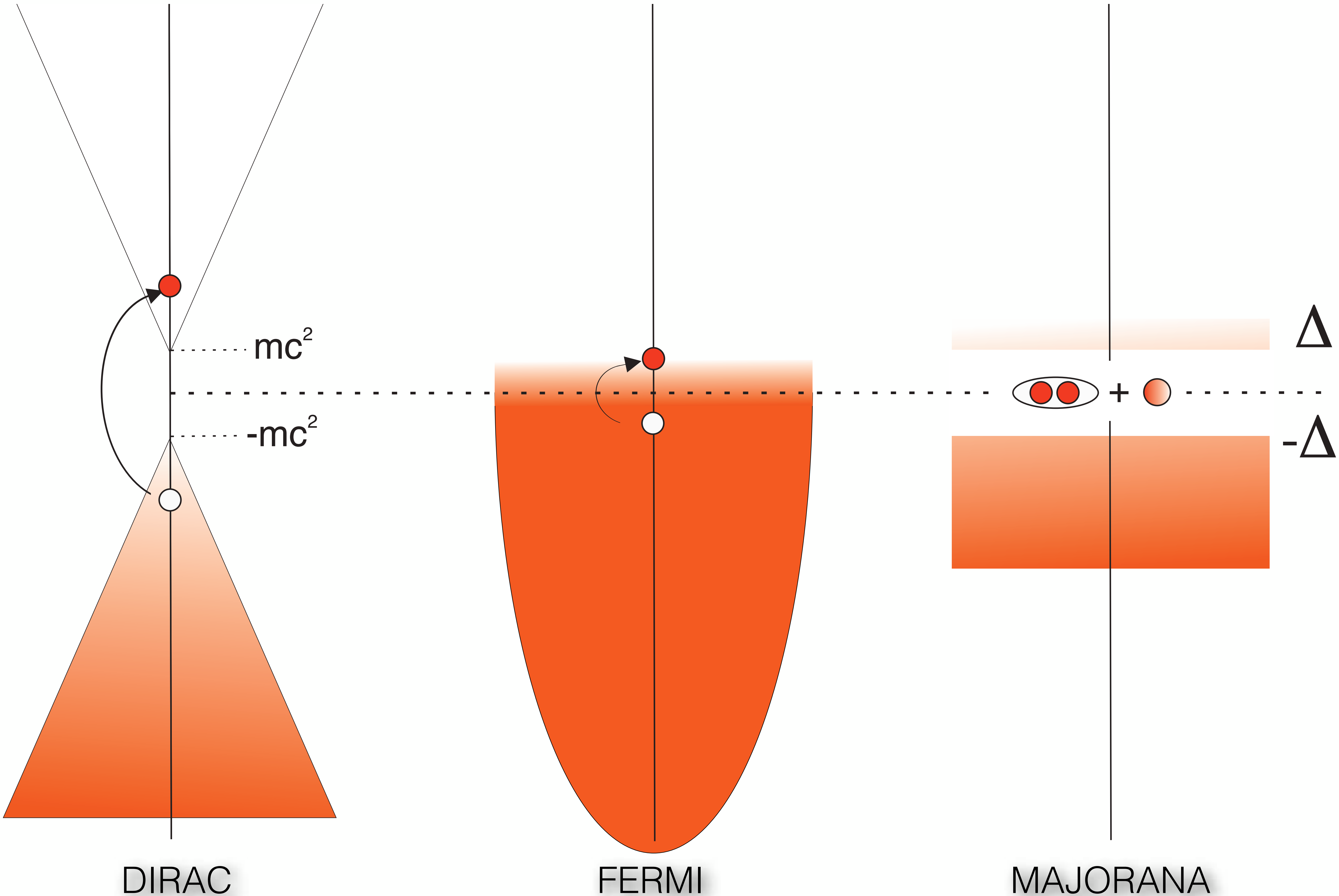
$$\gamma_0^\dagger = \gamma_0$$

Majorana Fermions in Superconductors

In condensed matter settings, spin degeneracy is a problem: we need **spinless superconductors** such as p-wave (early proposals). Intrinsic p-wave superconductors very scarce in nature: Sr_2RuO_4, (Matsumoto and Sigrist 1999), 5/2 fractional QHE (Read & Green, 2000), Kitaev 2001, etc



$$\gamma_0^\dagger = \gamma_0$$


mc2
-mc2
Δ
-Δ
+
DIRAC
FERMI
MAJORANA

Seminal result in field theory (Jackiw and Rossi, 1981): the Majorana equation in two dimensions has non-propagating solutions, namely **zero modes**, if the mass term has a **vortex** profile.

$$\begin{pmatrix} \sigma \cdot p & m \\ m & -\sigma \cdot p \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = i\partial_t \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

$$\sigma . p\psi + i m(\vec{r})\sigma_y \psi^* = E\psi$$

Vortex in 2D

For an introduction see: "Fractional and Majorana fermions", Jackiw, arXiv1104.4486

Dirac equation in Weyl representation (chiral spinors where the spin of a particle points in the same direction as its momentum)

$$\begin{pmatrix} -m & i\partial_t - \boldsymbol{\sigma}\mathbf{p} \\ i\partial_t + \boldsymbol{\sigma}\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

Impose charge conjugation $\Psi^C = \Psi$

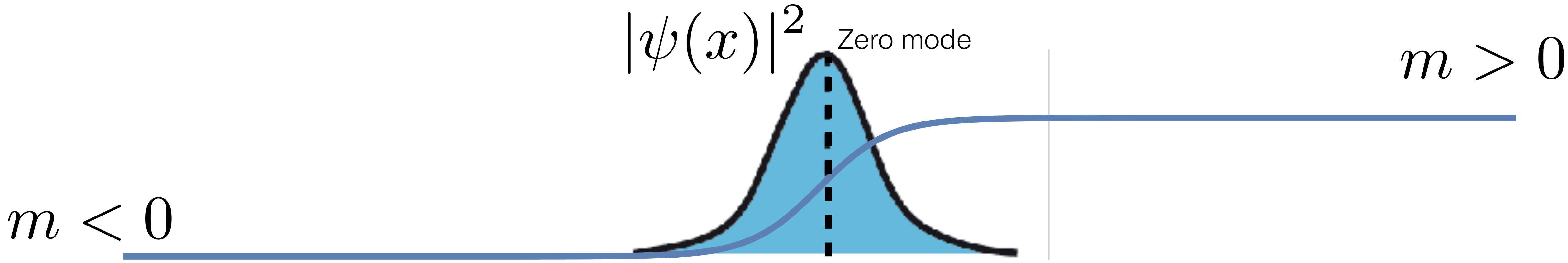
$$(i\partial_t - \boldsymbol{\sigma}\mathbf{p})\psi_R - im\sigma_y\psi_R^* = 0$$

$$(i\partial_t + \boldsymbol{\sigma}\mathbf{p})\psi_L - im\sigma_y\psi_L^* = 0$$

In the Majorana equation, chiralities become decoupled and **the mass term mixes particles and antiparticles** (technically, this implies that the equation is not gauge invariant, hence Majorana particles cannot be coupled to the electromagnetic field and are thus necessarily charge neutral).

This is a generalisation of another seminal result by Jackiw and Rebbi (Phys. Rev. D **13**, 3398, 1976) which demonstrates the existence of zero energy bound states in the one dimensional Dirac equation at points where **the mass term changes sign**.

$$[-i\hbar c\partial_x\sigma_x + m(x)c^2\sigma_z]\psi(x) = E\psi(x)$$



For an introduction see: “Fractional and Majorana fermions”, Jackiw, arXiv1104.4486

Two-dimensional chiral p+ip topological superconductor

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{BdG} = \begin{pmatrix} \frac{p_x^2+p_y^2}{2m} - \mu & -i\Delta(p_x - ip_y) \\ i\Delta^*(p_x + ip_y) & -\frac{p_x^2+p_y^2}{2m} + \mu \end{pmatrix}$$

Low momentum

$$H_{BdG} = \begin{pmatrix} -\mu & -i\Delta(p_x - ip_y) \\ i\Delta^*(p_x + ip_y) & \mu \end{pmatrix}$$

N. Read and D. Green, "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect". Phys. Rev. B **61**, 10267 (2000)

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{BdG} = \begin{pmatrix} -\mu & -i\Delta(p_x - ip_y) \\ i\Delta^*(p_x + ip_y) & \mu \end{pmatrix}$$

$$H_{BdG}(p_y = 0) = \Delta p_x \tau^y - \mu \tau^z$$

One-dimensional massive Dirac Hamiltonian (with Pauli matrices defined in electron-hole space). As discussed by Jackiw and Rebbi (Phys. Rev. D **13**, 3398, 1976), this one-dimensional Dirac Hamiltonian contains **zero energy bound states** at interfaces where the mass term changes sign (which is our case as we cross the $\mu = 0$ critical point)

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{BdG} = \begin{pmatrix} -\mu(x) & -i\Delta(-i\partial_x - ip_y) \\ i\Delta^*(i\partial_x + ip_y) & \mu(x) \end{pmatrix}$$

$$\mu(x) = \begin{cases} -\mu_0 < 0, & \text{for } x < 0 \\ \mu_0 > 0, & \text{for } x > 0 \end{cases}$$

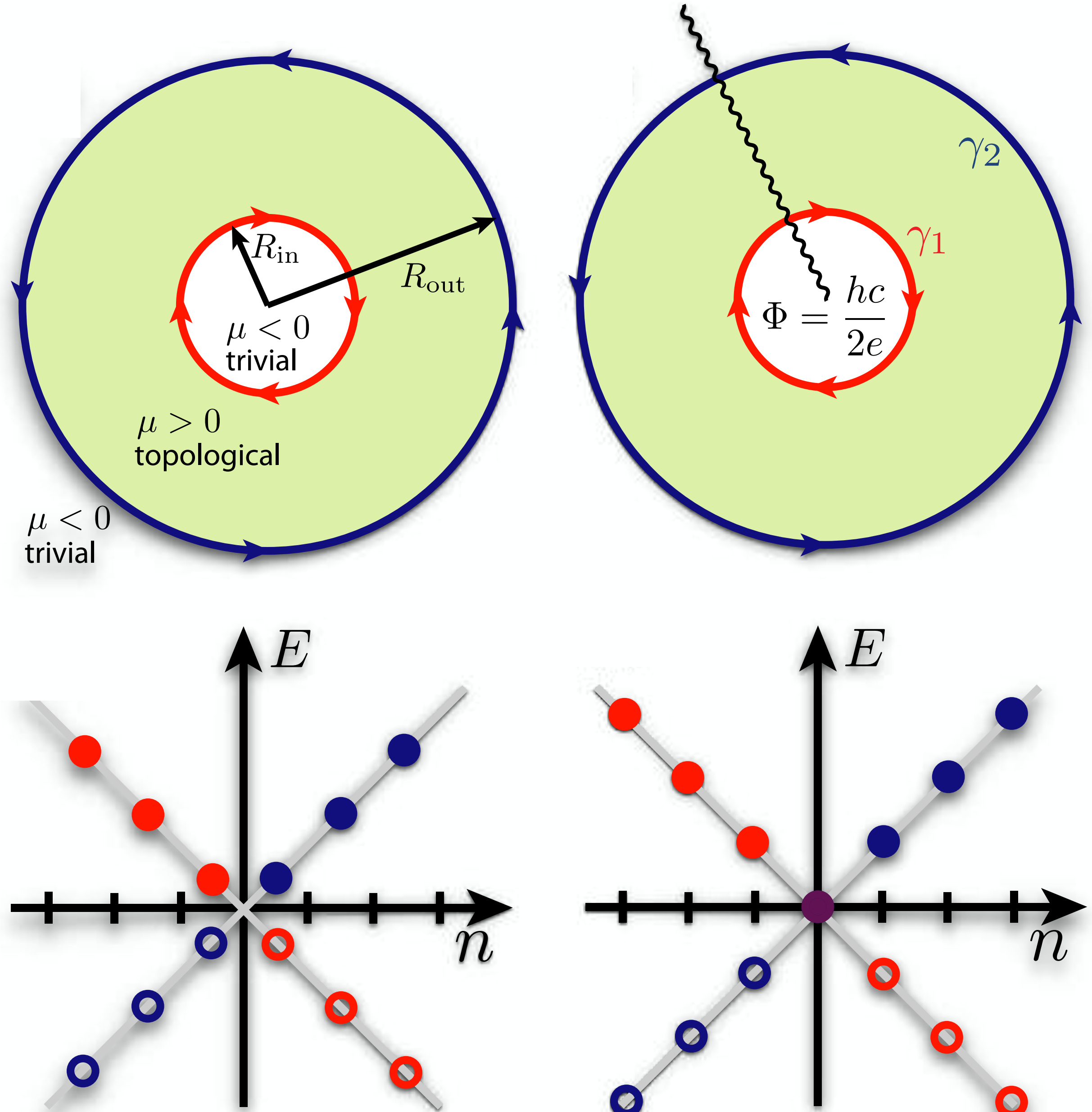
$$\Phi(x) \sim \exp\left[-\int_0^x dx' \frac{\mu(x')}{\Delta}\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\gamma \sim \int dx \exp\left[-\int_0^x dx' \frac{\mu(x')}{\Delta}\right] [c(x) + c^\dagger(x)]$$

Zero energy bound states

Two-dimensional chiral p+ip topological superconductor

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor



Majorana zero modes bound to vortices

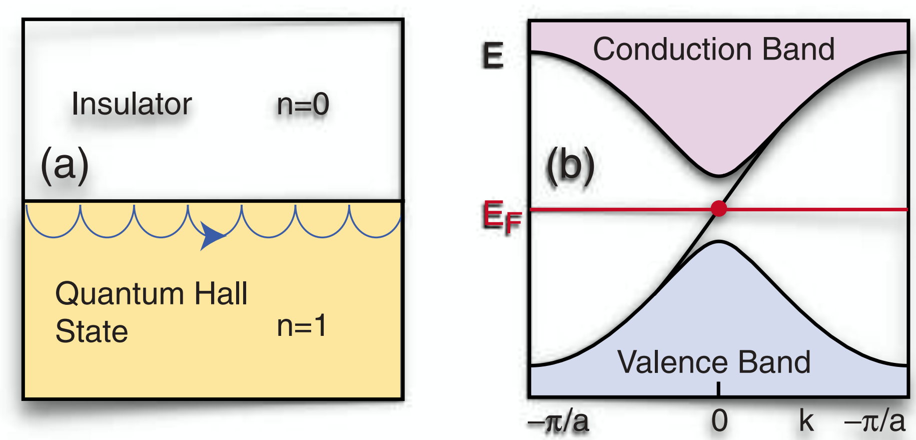
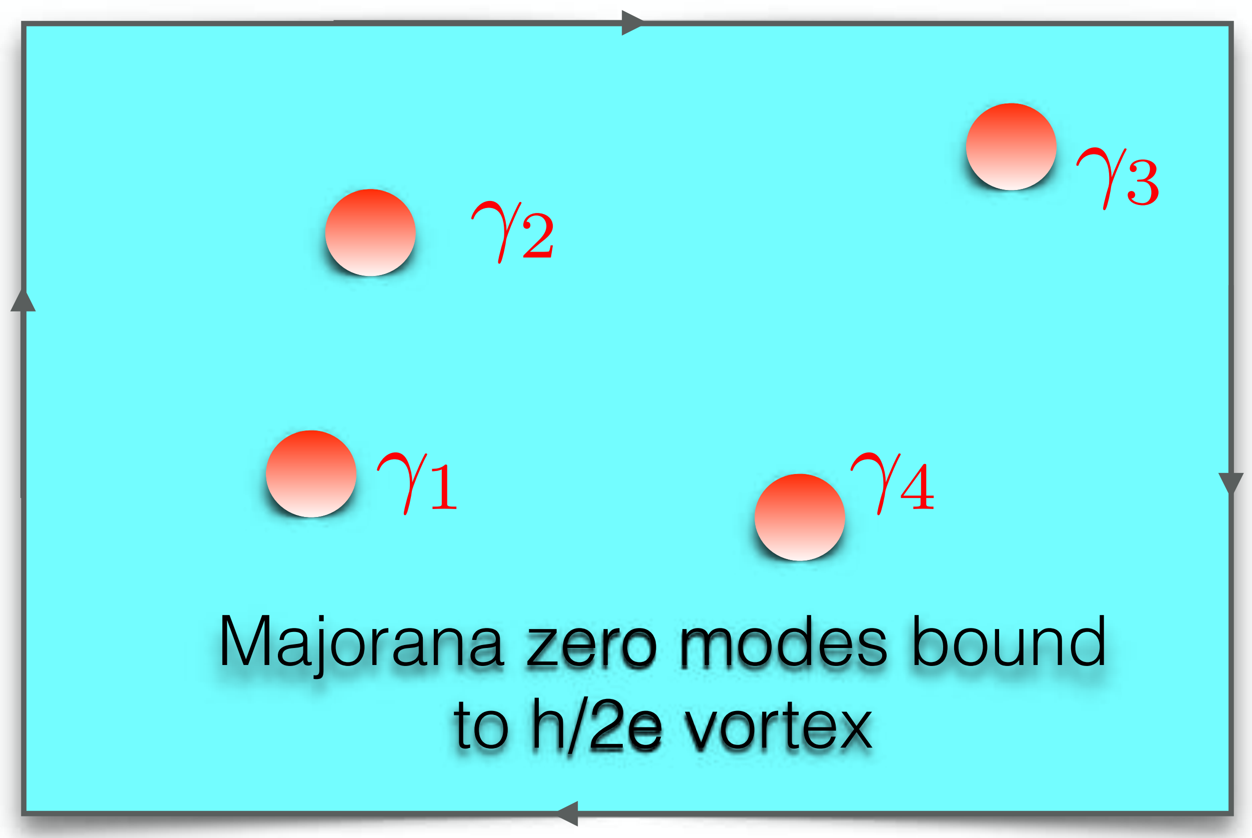


Image from J. Alicea, *"New directions in the pursuit of Majorana fermions in solid state systems"* Rep. Prog. Phys. 75, 076501(2012).

First proposals more than 20-30 years old!

Two-dimensional p-wave superconductor

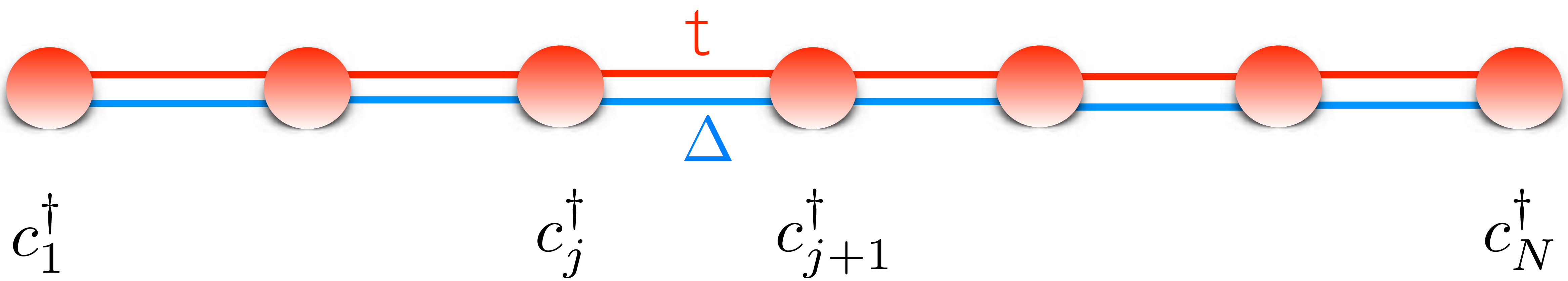


N. Read and D. Green, "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect". Phys. Rev. B **61**, 10267 (2000)

Kitaev, Phys.-Usp., **44**, 131, 2001

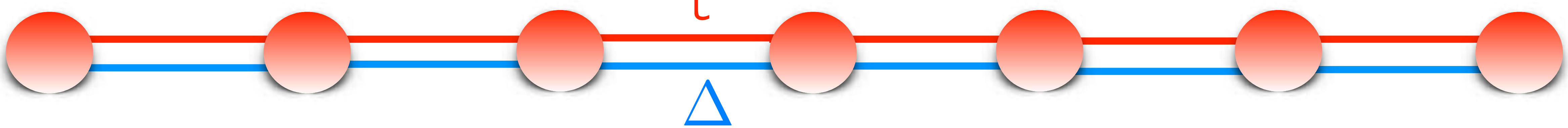
$$H = -\mu \sum_{j=1}^{N} \left(c_j^\dagger c_j - \frac{1}{2} \right) + \sum_{j=1}^{N-1} \left[-t \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \Delta \left(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger \right) \right],$$

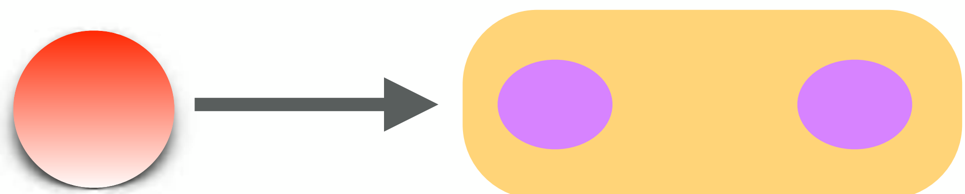
1d spinless chain with p-wave pairing



The Kitaev model

Kitaev, Phys.-Usp., **44**, 131, 2001





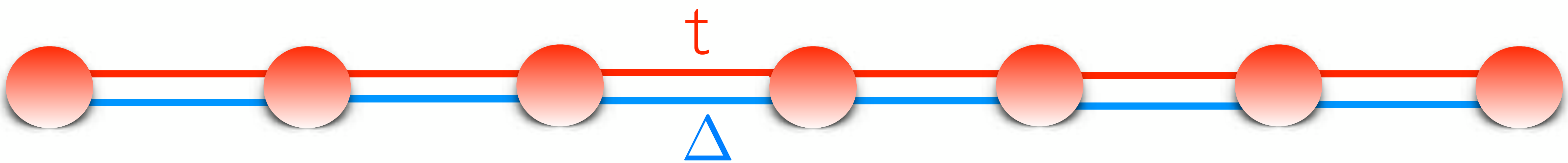
Decomposition in terms of Majorana operators

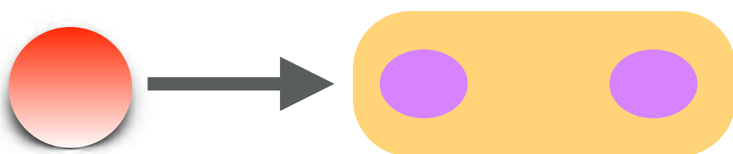
$$c_j = \frac{1}{2}\left(\gamma_j^A + i\gamma_j^B\right), \quad c_j^\dagger = \frac{1}{2}\left(\gamma_j^A - i\gamma_j^B\right).$$

$$\{\gamma_i^A, \gamma_j^B\} = 2\delta_{ij}\delta_{AB}\,, \quad \gamma_j = \gamma_j^\dagger\,, \quad \gamma_j^2 = \gamma_j^{\dagger 2} = 1$$

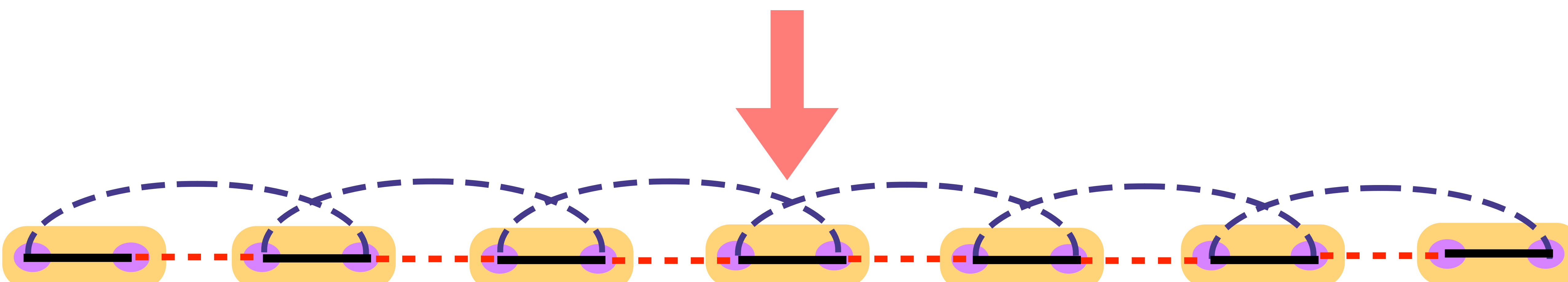
The Kitaev model

Kitaev, Phys.-Usp., **44**, 131, 2001





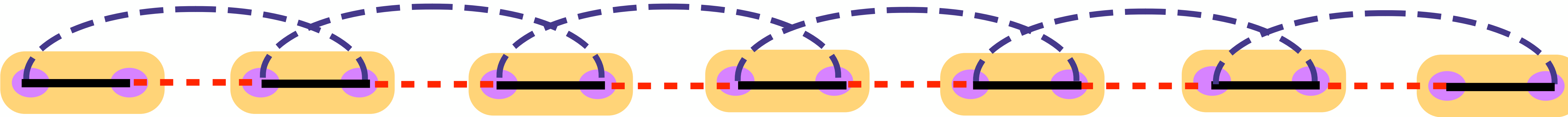
$$\gamma_j^A = c_j + c_j^\dagger \,, \qquad \gamma_j^B = i(c^\dagger - c_j)$$



$$\gamma_j^A \quad \gamma_j^B \quad \gamma_{j+1}^A \; \gamma_{j+1}^B$$

The Kitaev model

Kitaev, Phys.-Usp., **44**, 131, 2001



$\gamma_j^A \quad \gamma_j^B \quad \gamma_{j+1}^A \quad \gamma_{j+1}^B$

$$H = -\mu \sum_{j=1}^{N} \left(c_j^\dagger c_j - \frac{1}{2} \right) + \sum_{j=1}^{N-1} \left[-t \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \Delta \left(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger \right) \right],$$

$$H = -\frac{i\mu}{2} \sum_{j=1}^{N} \gamma_j^A \gamma_j^B + \frac{i}{2} \sum_{j=1}^{N-1} \left[\omega_+ \gamma_j^B \gamma_{j+1}^A + \omega_- \gamma_j^A \gamma_{j+1}^B \right],$$

$\omega_+ = \Delta + t$

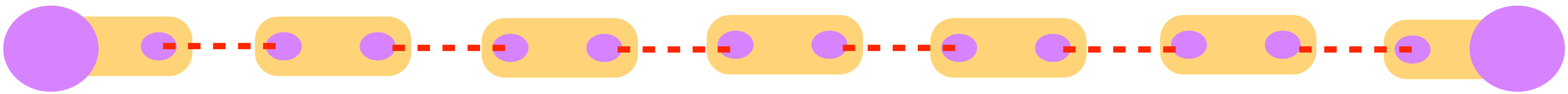
$\omega_- = \Delta - t$

μ

The Kitaev model

Kitaev, Phys.-Usp., **44**, 131, 2001

$$\mu = 0 \quad t = \Delta$$



$$H = -\frac{i\mu}{2}\sum_{j=1}^{N}\gamma_j^A\gamma_j^B + \frac{i}{2}\sum_{j=1}^{N-1}\left[\omega_+\gamma_j^B\gamma_{j+1}^A + \omega_-\gamma_j^A\gamma_{j+1}^B\right],$$

$$H = it\sum_{j=1}^{N-1}\gamma_j^B\gamma_{j+1}^A \,.$$

Despite its innocent-looking form, this model is rather nontrivial:

- Majorana operators on the same site are now decoupled
- long range coupling is established since Majorana operators on neighbouring sites are now coupled.
- The Majorana operators at the end of chain seem to have disappeared from the problem…

$$\omega_+ = 2t$$

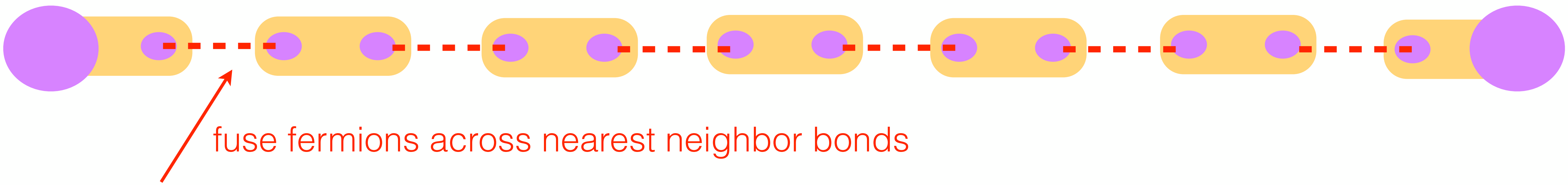
$$\omega_- = 0$$

$$\mu = 0$$

The Kitaev model

Kitaev, Phys.-Usp., **44**, 131, 2001

$$\mu = 0 \quad t = \Delta$$



$$d_j = \frac{1}{2}\left(\gamma_j^B + i\gamma_{j+1}^A\right)$$

$$d_j^\dagger = \frac{1}{2}\left(\gamma_j^B - i\gamma_{j+1}^A\right).$$

$$H = 2t \sum_{j=1}^{N-1} \left(d_j^\dagger d_j - \frac{1}{2}\right).$$

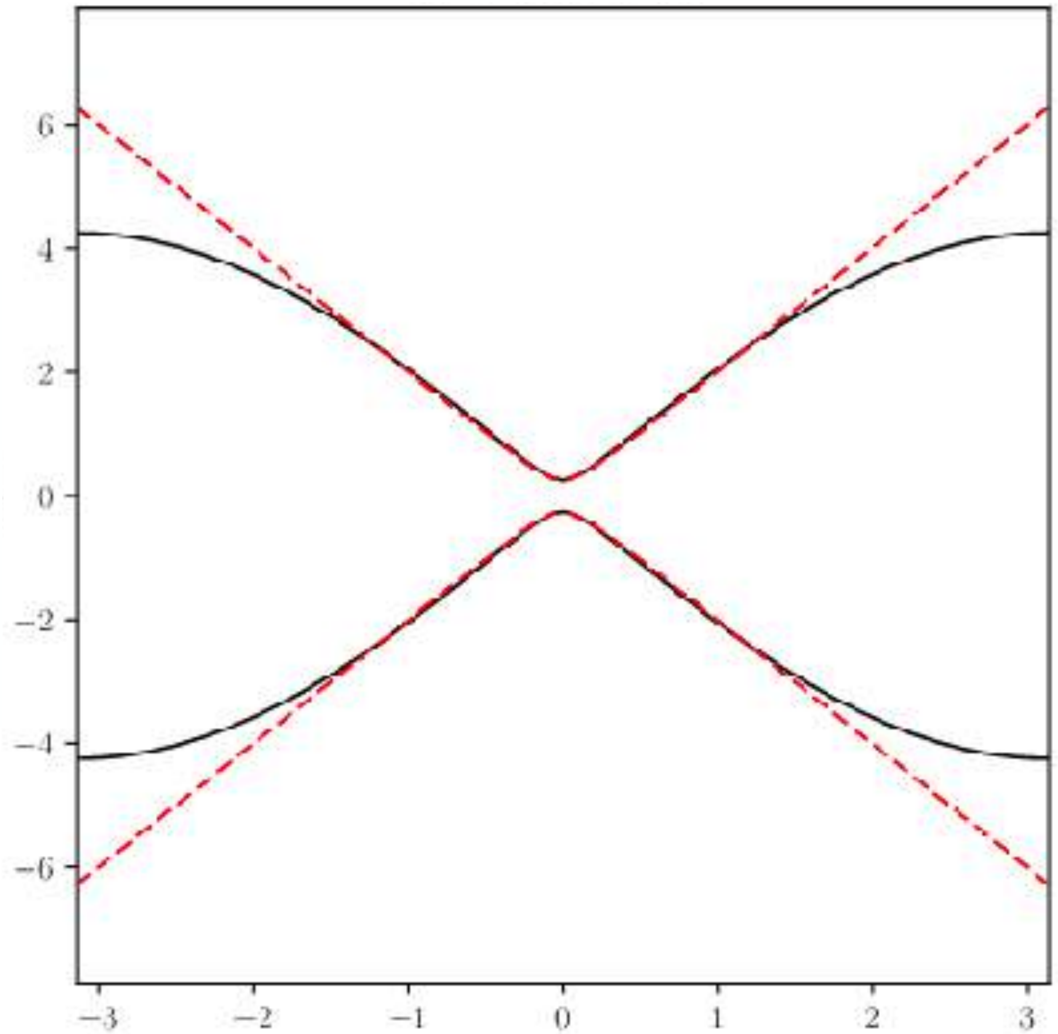
$$[H, \gamma_1^A] = [H, \gamma_N^B] = 0.$$

GAPPED SPECTRUM+ZERO-ENERGY MAJORANAS AT THE END OF THE WIRE (DECOUPLED FROM THE BULK OF THE CHAIN)!!!

Kitaev, Phys.-Usp., **44**, 131, 2001

$\mu = -2.25t$ Trivial Phase

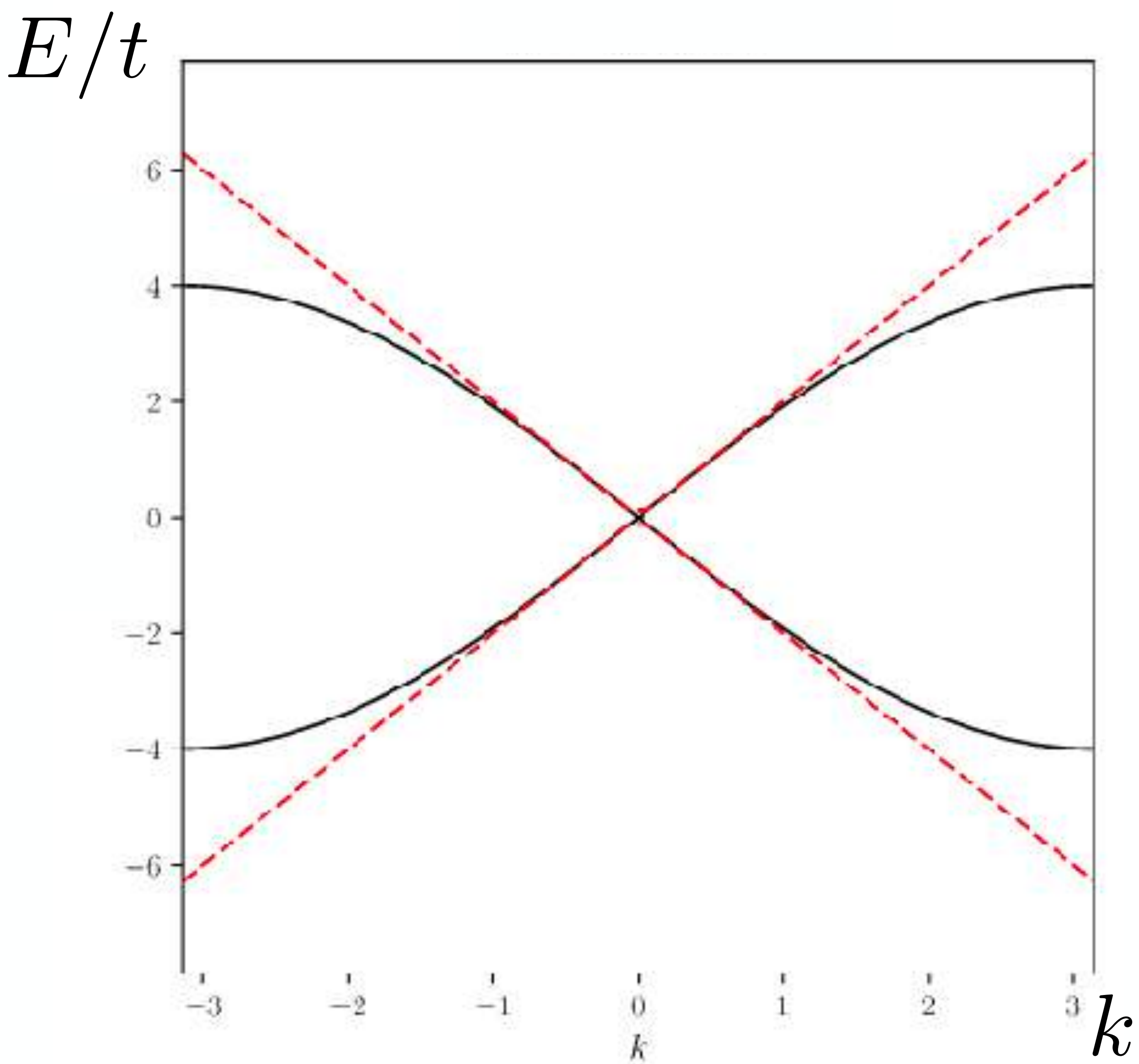
E/t



k

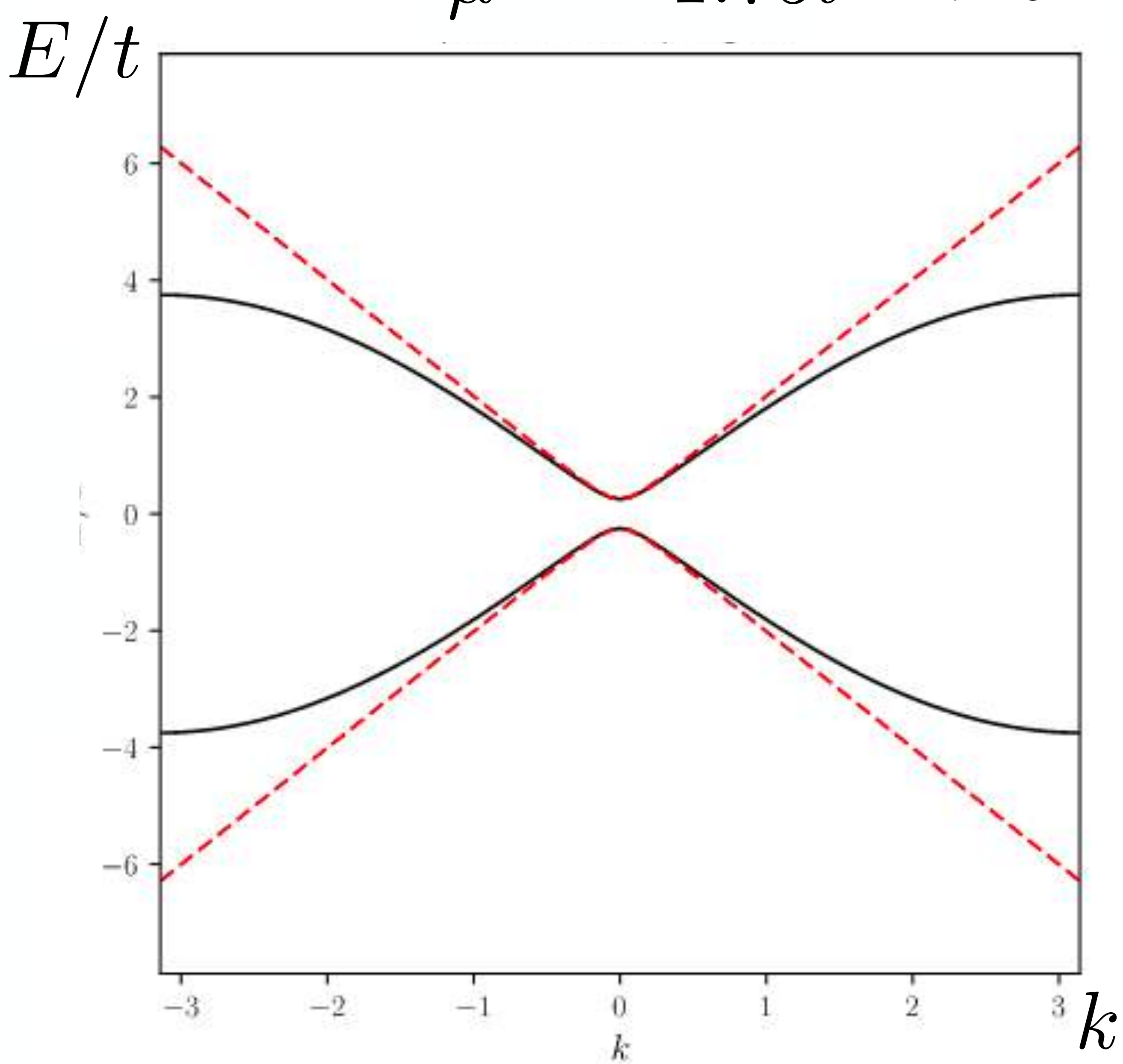
Kitaev, Phys.-Usp., **44**, 131, 2001

$\mu = -2t$ **Gap closes**

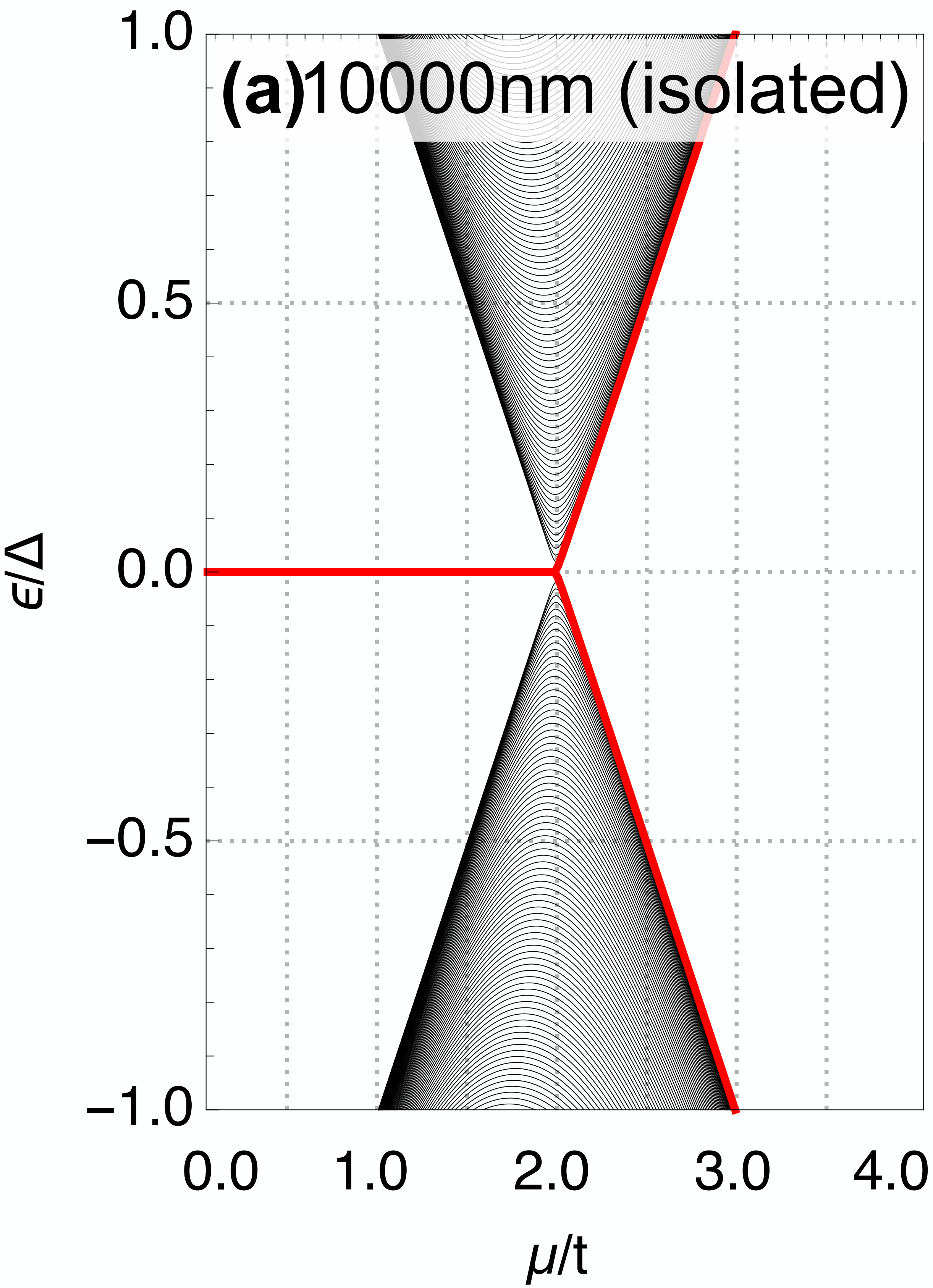


Kitaev, Phys.-Usp., **44**, 131, 2001

$\mu = -1.75t$ Topological Phase



Kitaev, Phys.-Usp., **44**, 131, 2001

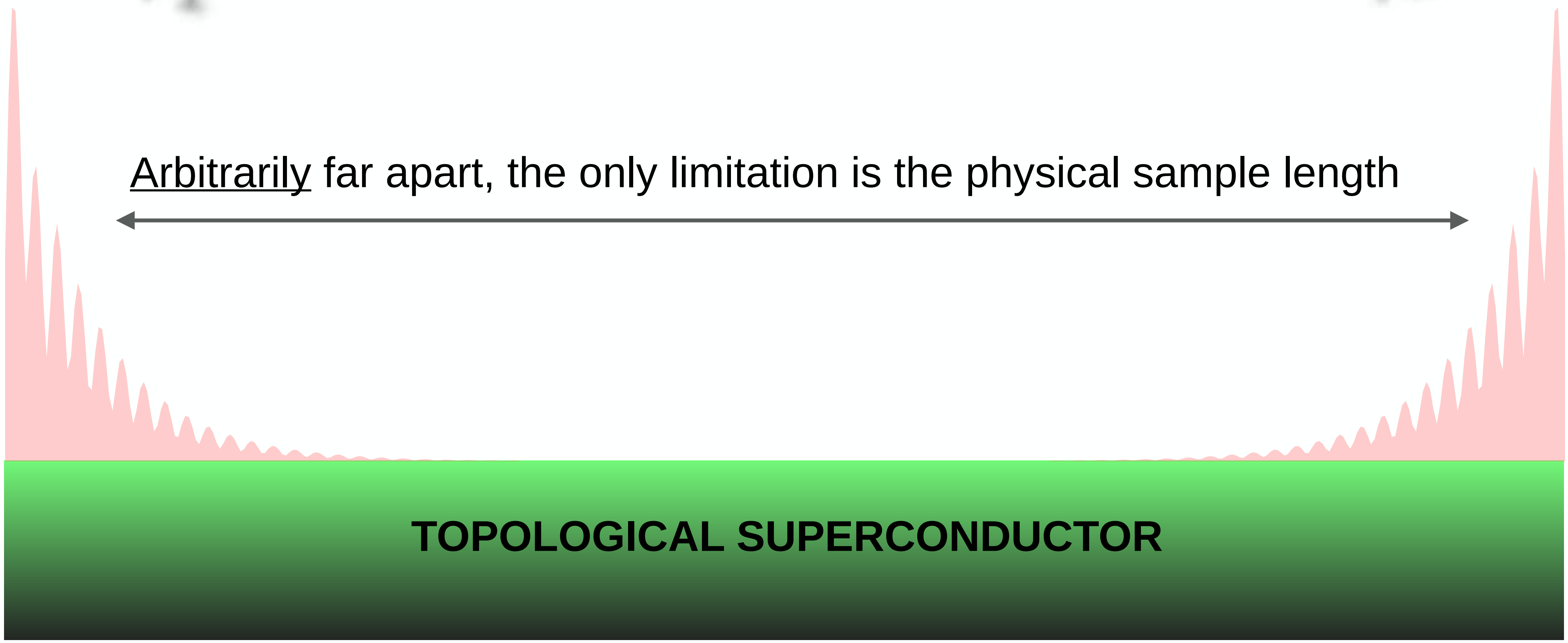


The system is topologically non-trivial (topological superconductor) for

$$|\mu| < 2t$$

MAJORANA STATES

$$\gamma_1 = \gamma_1^\dagger \qquad \gamma_2 = \gamma_2^\dagger$$



Two Majorana quasiparticles at the ends of a one dimensional topological superconductor form a single electronic excitation which can be arbitrarily delocalized.

$$-i\gamma_1\gamma_2 = (1 - 2c^\dagger c)$$

Statistics under exchange

$$|\Psi\rangle \rightarrow |\Psi\rangle$$

Bosons

$$|\Psi\rangle \rightarrow -|\Psi\rangle$$

Fermions

CSIC

- Phase factors: the order of the exchange is irrelevant
- Example 1/3 fractional QHE

$$|\Psi\rangle \rightarrow e^{i\theta}|\Psi\rangle$$

Abelian Anyons

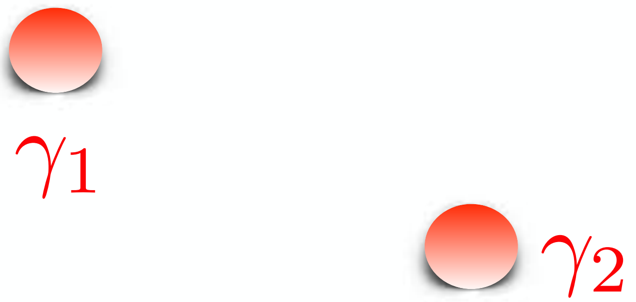
J. Leinaas, J. Myrheim, Jan, "On the theory of identical particles" *Il Nuovo Cimento B*. **37,** 1, 1977
Frank Wilczek, “Quantum Mechanics of Fractional-Spin Particles”, Phys. Rev. Lett. **49**, 957, 1982

CSIC

- Braid matrix: the order of the exchange matters
- After the exchange, the ground state **<u>changes</u>**

$$|\Psi_i\rangle \rightarrow U_{ij}|\Psi_j\rangle$$

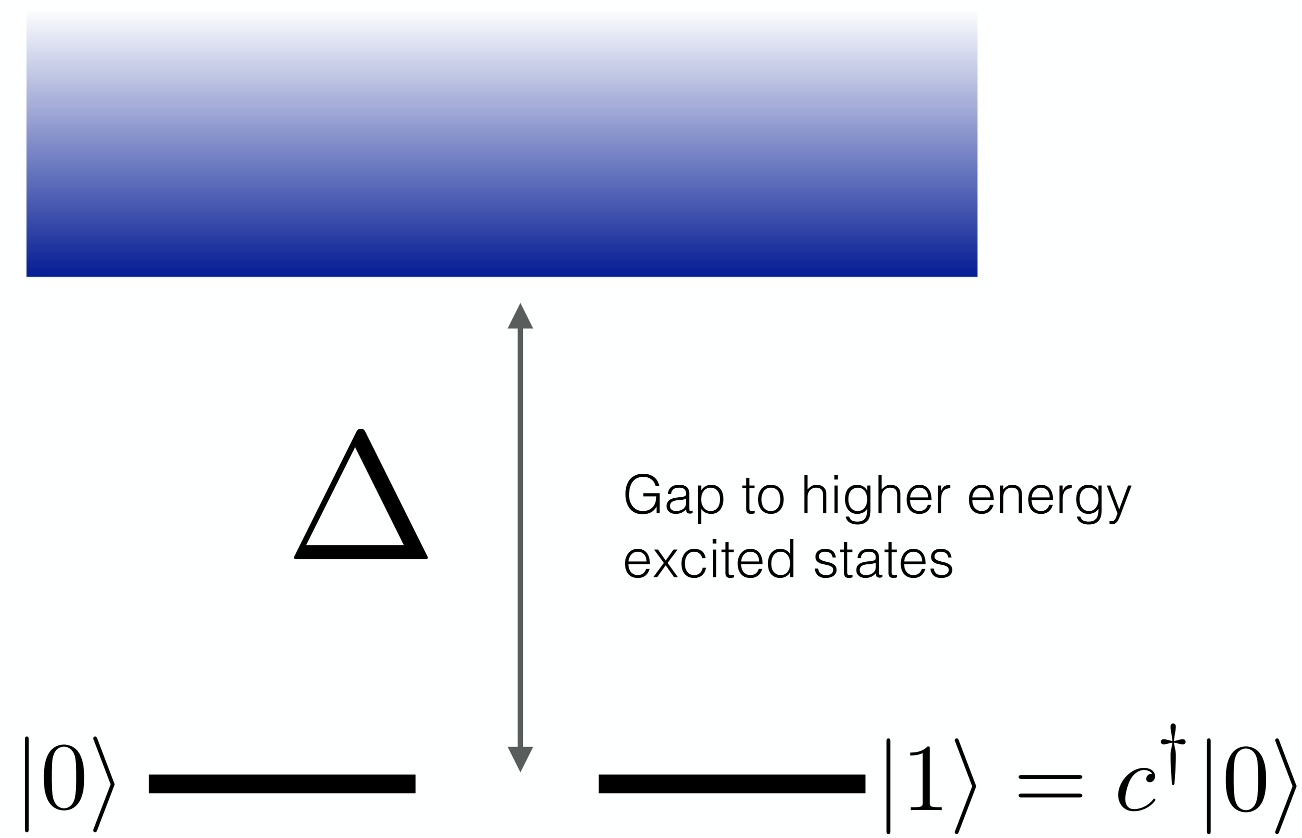
non-Abelian Anyons



Fermionic parity operator

$$P = 1 - 2\hat{n} = 1 - 2c^\dagger c = -i\gamma_1\gamma_2$$

Majorana zero modes encode ground state degeneracy



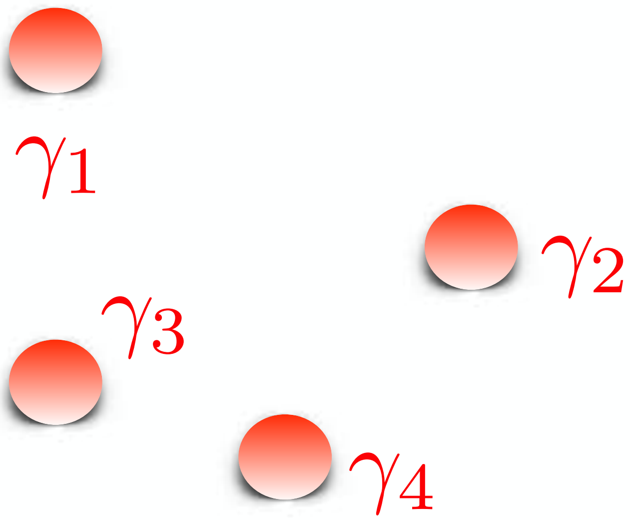
Majorana zero mode is "half" a Fermion

$$c = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

$$c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

Locally indistinguishable ground states

Two fermionic parity ground states

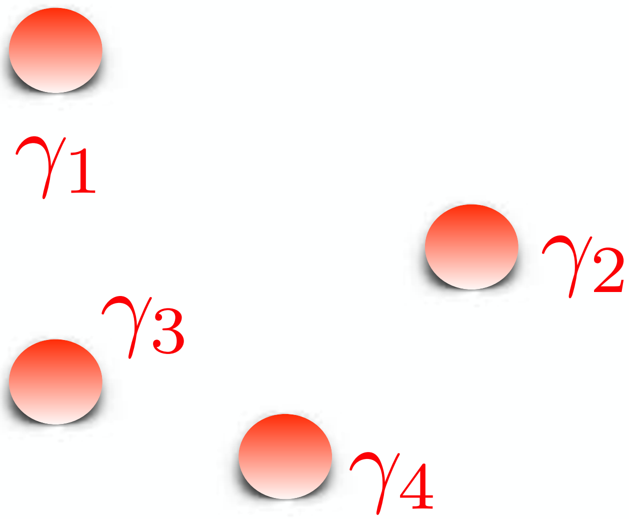


$$c_A = \frac{1}{2}(\gamma_1 + i\gamma_2) \qquad c_B = \frac{1}{2}(\gamma_3 + i\gamma_4)$$

$$c_A^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2) \qquad c_B^\dagger = \frac{1}{2}(\gamma_3 - i\gamma_4)$$

Two fermionic parity ground states	
$\vert 0,0\rangle$ **Even**	$\vert 0,1\rangle = c_B^\dagger \vert 0,0\rangle$ **Odd**
$\vert 1,1\rangle = c_A^\dagger c_B^\dagger \vert 0,0\rangle$	$\vert 1,0\rangle = c_A^\dagger \vert 0,0\rangle$
$\vert\uparrow\rangle \equiv \vert 0,0\rangle$	$\vert\uparrow\rangle \equiv \vert 1,0\rangle$
$\vert\downarrow\rangle \equiv \vert 1,1\rangle$	$\vert\downarrow\rangle \equiv \vert 0,1\rangle$

Majorana zero modes as building blocks of topological qubits



Two fermionic parity ground states	
$\|0,0\rangle$ **Even**	$\|0,1\rangle = c_B^\dagger \|0,0\rangle$ **Odd**
$\|1,1\rangle = c_A^\dagger c_B^\dagger \|0,0\rangle$	$\|1,0\rangle = c_A^\dagger \|0,0\rangle$
$\|\uparrow\rangle \equiv \|0,0\rangle$	$\|\uparrow\rangle \equiv \|1,0\rangle$
$\|\downarrow\rangle \equiv \|1,1\rangle$	$\|\downarrow\rangle \equiv \|0,1\rangle$

We can define a Majorana qubit by fixing the parity (even or odd) of the total number of fermions and construct the three Pauli matrices using four Majorana zero modes.

$$\sigma_x = -i\gamma_2\gamma_3$$

$$\sigma_y = -i\gamma_1\gamma_3$$

$$\sigma_z = -i\gamma_1\gamma_2,$$

The qubit stores quantum information non-locally (through the spatially separated Majorana zero modes) which gives rise to the concept of topologically protected (fault-tolerant) quantum computation (inmune to local errors/decoherence)

$$U_{12} = e^{-i\frac{\pi}{4}\gamma_1\gamma_2}$$

$$\gamma_1 \rightarrow -\gamma_2$$

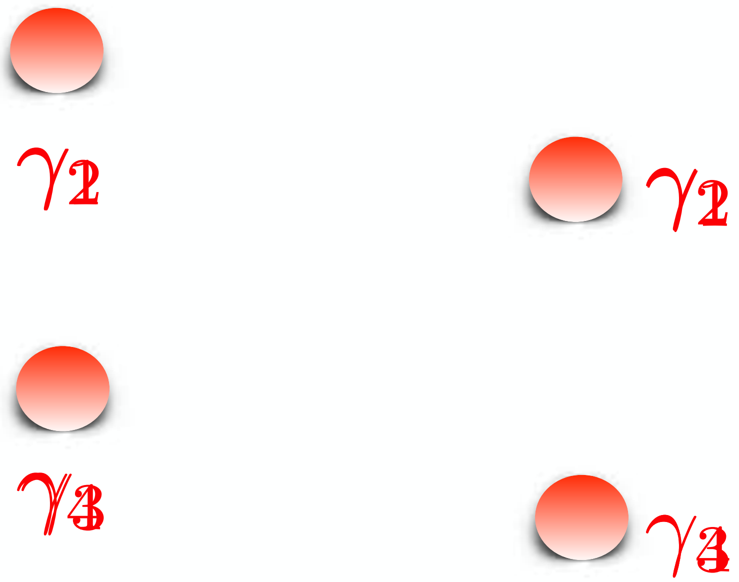
$$\gamma_2 \rightarrow \gamma_1$$

$$U_{12}|n\rangle = e^{\frac{\pi}{4}(1-2n)}|n\rangle$$

Exchanging two Majorana zero modes generates occupancy dependent phase factors

$$U_{12}|0\rangle = \frac{1}{\sqrt{2}}(1+i)|0\rangle \qquad U_{12}|1\rangle = \frac{1}{\sqrt{2}}(1-i)|1\rangle$$

Majorana zero modes encode ground state degeneracy

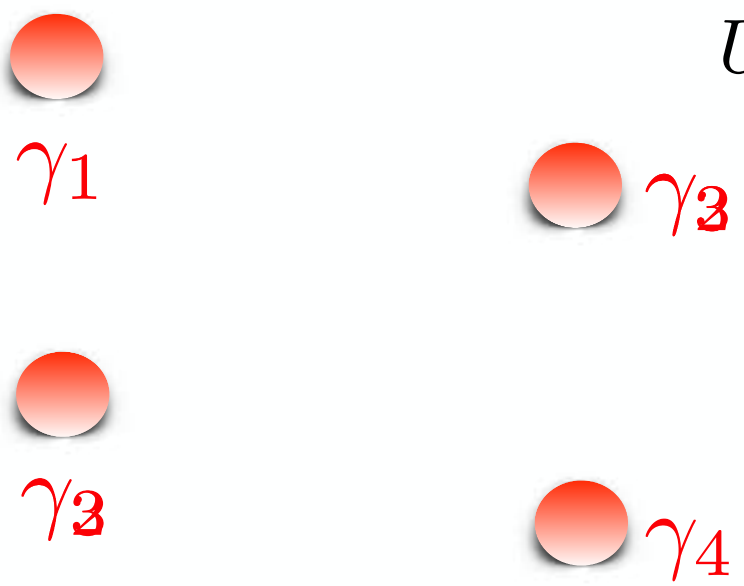


Two fermionic parity ground states	
$\|0,0\rangle$ **Even**	$\|0,1\rangle = c_B^\dagger\|0,0\rangle$ **Odd**
$\|1,1\rangle = c_A^\dagger c_B^\dagger\|0,0\rangle$	$\|1,0\rangle = c_A^\dagger\|0,0\rangle$
$\|\uparrow\rangle \equiv \|0,0\rangle$	$\|\uparrow\rangle \equiv \|1,0\rangle$
$\|\downarrow\rangle \equiv \|1,1\rangle$	$\|\downarrow\rangle \equiv \|0,1\rangle$

$$U_{12}|n_1,n_2\rangle = e^{\frac{\pi}{4}(1-2n_1)}|n_1,n_2\rangle$$
$$U_{34}|n_1,n_2\rangle = e^{\frac{\pi}{4}(1-2n_2)}|n_1,n_2\rangle,$$

Occupation-dependent phase factors

Non-Abelian braiding of Majorana zero modes



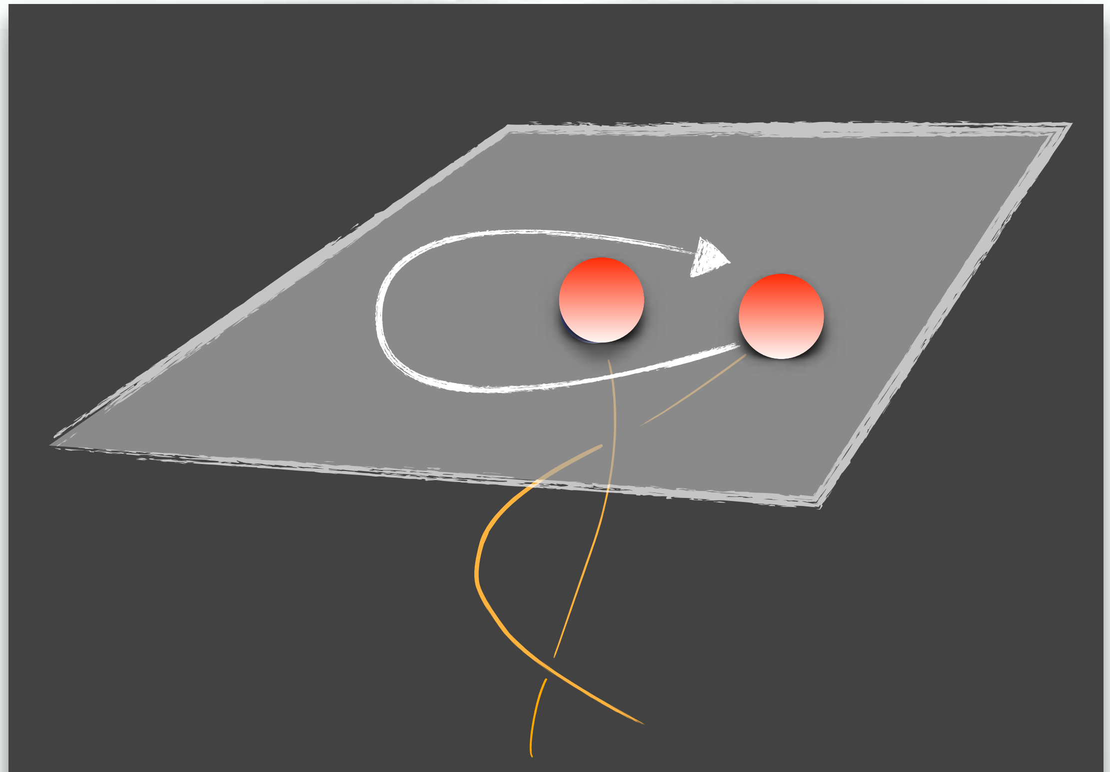
$$U_{23}|n_1, n_2\rangle = \frac{1}{\sqrt{2}}\left[|n_1, n_2\rangle + i(-1)^{n_1}|1-n_1, 1-n_2\rangle\right].$$

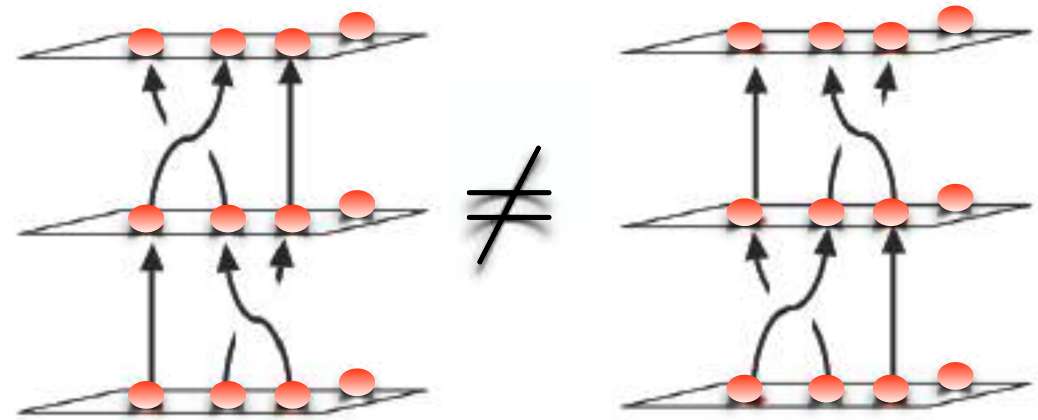
$$U_{23}|0,0\rangle = \frac{1}{\sqrt{2}}\left[|0,0\rangle + i|1,1\rangle\right].$$

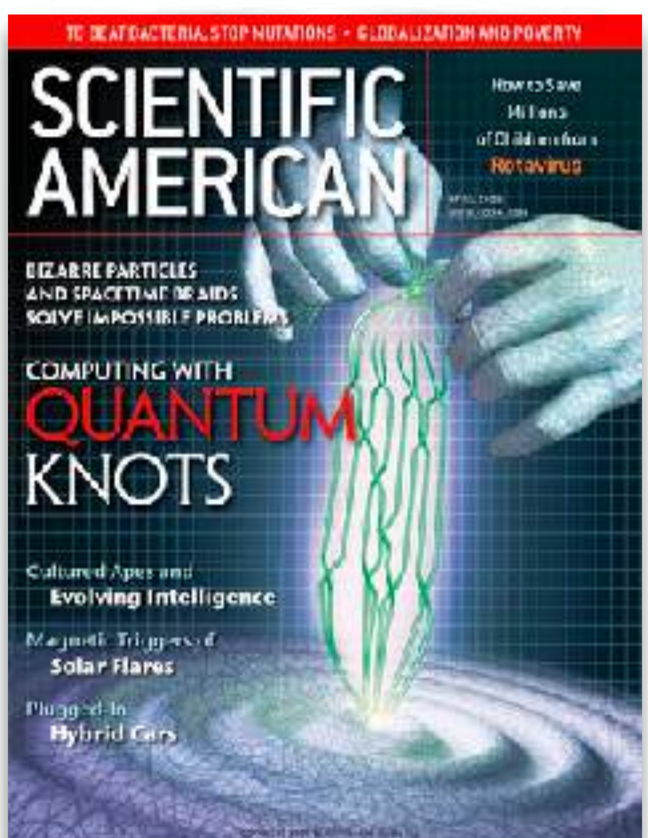
$$U_{23}|1,1\rangle = \frac{1}{\sqrt{2}}\left[|1,1\rangle - i|0,0\rangle\right].$$

$$[U_{n-1,n}U_{n,n+1}] = i\gamma_{n-1}\gamma_{n+1}.$$

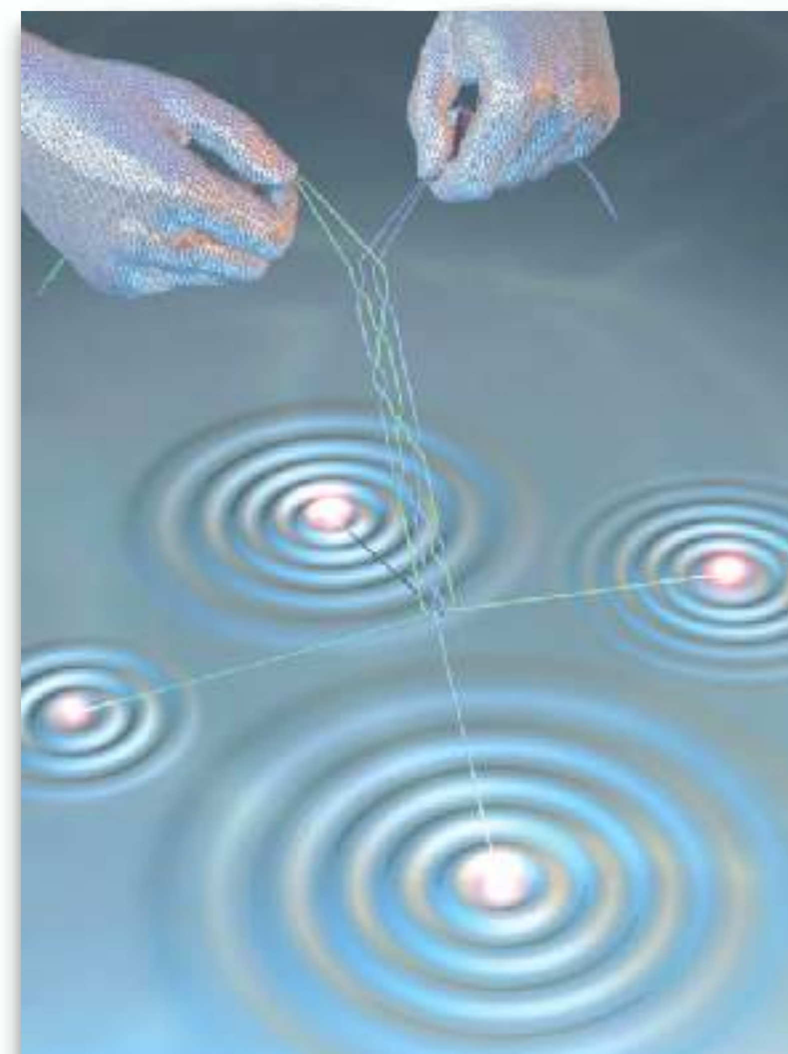
- Majorana exchange produces a superposition state of **different number states.**
- Whenever two exchanges involve the same Majorana zero mode, the braid operators do not conmute.
- The final state of the system after exchanging several pairs of Majorana zero modes depends on the order of the exchange operations, much like braiding cords in a necklace, hence the name non-Abelian "braiding"







- *Non-Abelian anyons and topological quantum computation*, Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma, Rev. Mod. Phys. 80, 1083, 2008
- *Non-Abelian states of matter*, Ady Stern,Nature 464, 187, 2010
- Majorana zero modes *and topological quantum computation*, Sankar Das Sarma, Michael Freedman and Chetan Nayak, Npj Quantum Information, (2015) 15001.



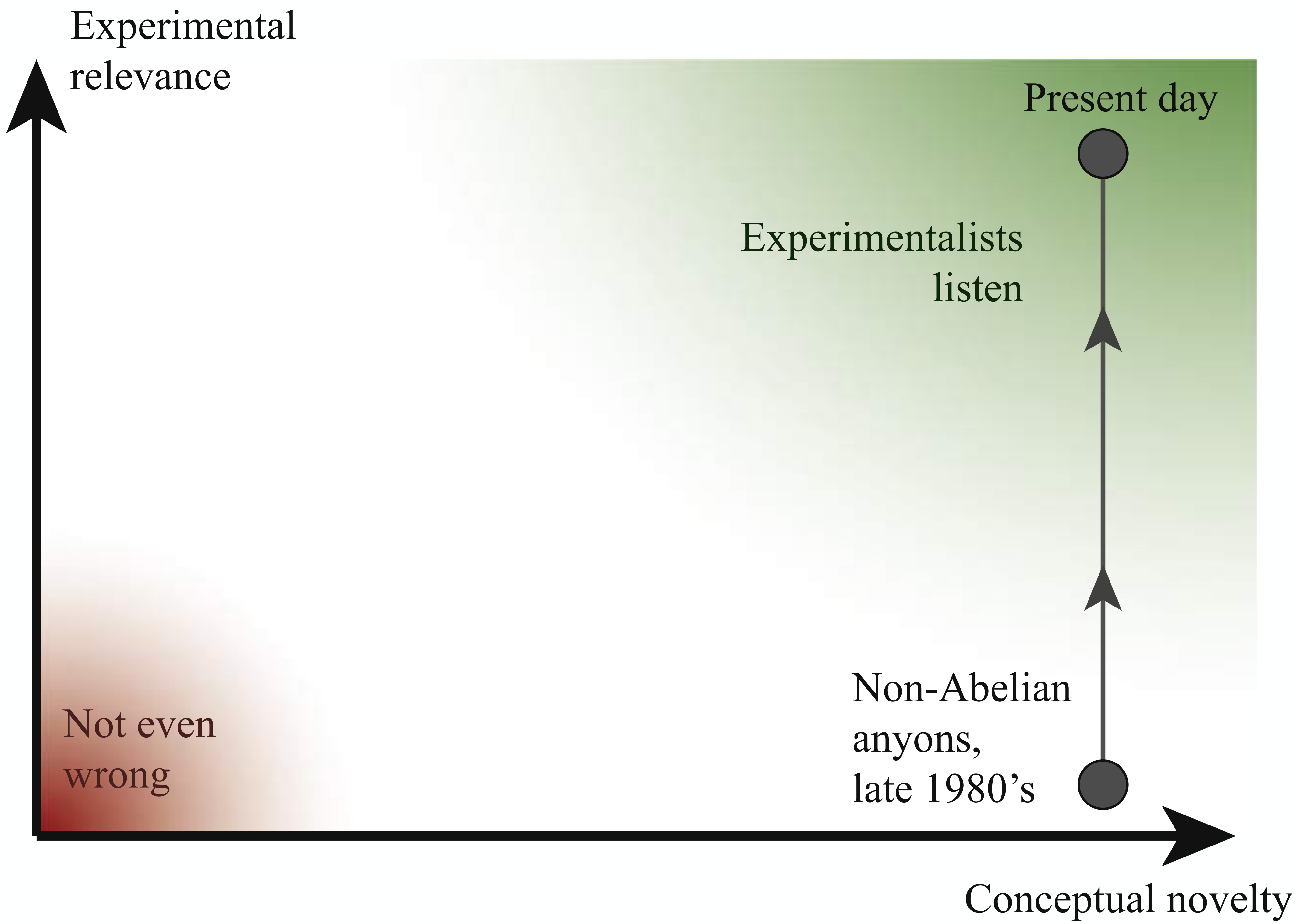
Challenging experimentally:

- electrons are spin-degenerate so we must freeze out half of the degrees of freedom to have an effective spinless system.
- p-wave superconductors are very rare in nature.

Clever proposals that overcome these challenges have the same three main ingredients:

1. Instead of using intrinsic superconductivity use the superconducting proximity effect.
2. Time-reversal symmetry breaking
3. Spin-orbit coupling.

Allows to use standard materials!!!



Jason Alicea and Ady Stern, "Designer non-Abelian anyon platforms: from Majorana to Fibonacci", Physica. Scripta. **2015**, 014006, 2015