

Emergence of Quantum Phases in Novel Materials

Instituto de Ciencia de Materiales de Madrid

Quantum Materials for Quantum Technologies (Q4Q)

Phase transitions

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More Is Different

Broken symmetry and the nature of
the hierarchical structure of science.

P. W. Anderson

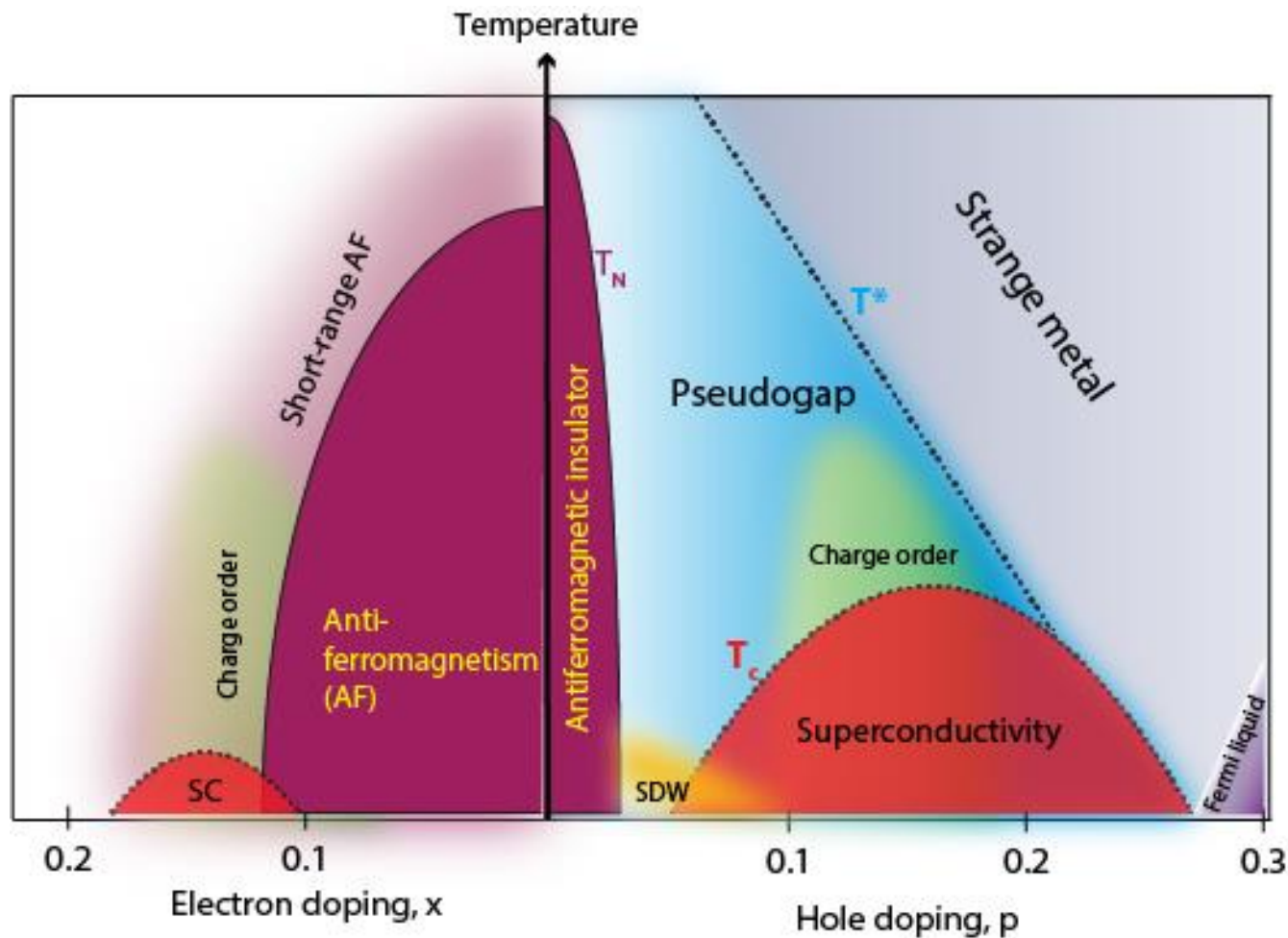
P. Anderson, Science'72

X	Y
solid state or many-body physics	elementary particle physics
chemistry	many-body physics
molecular biology	chemistry
cell biology	molecular biology
•	•
•	•
•	•
psychology	physiology
social sciences	psychology

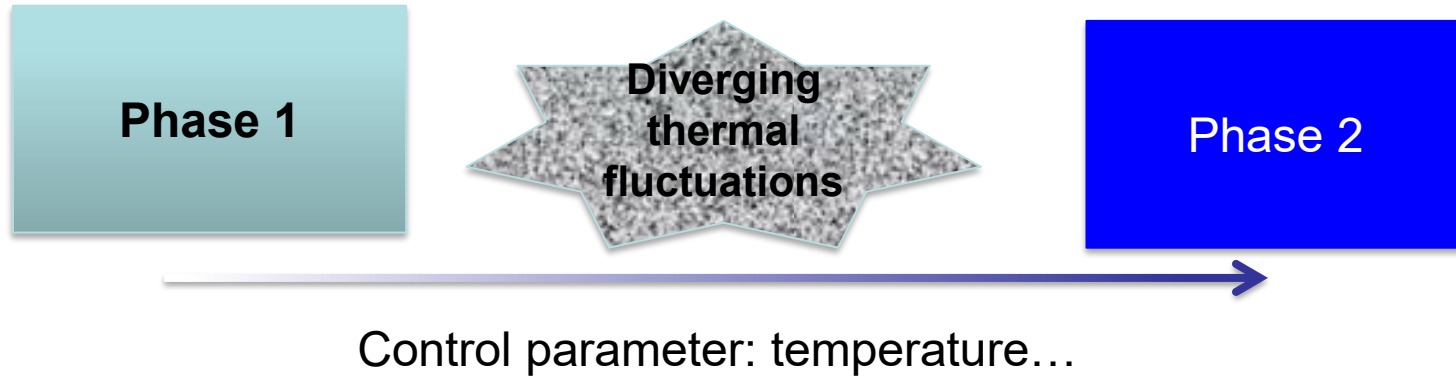
But this hierarchy does not imply that science X is “just applied Y.” At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry.

In my own field of many-body physics, we are, perhaps, closer to our fundamental, intensive underpinnings than in any other science in which non-trivial complexities occur, and as a result we have begun to formulate a general theory of just how this shift from quantitative to qualitative differentiation takes place. This formulation, called the theory of “broken symmetry,” may be of help in making more generally clear the breakdown of the constructionist converse of reductionism. I will give an elementary and in-

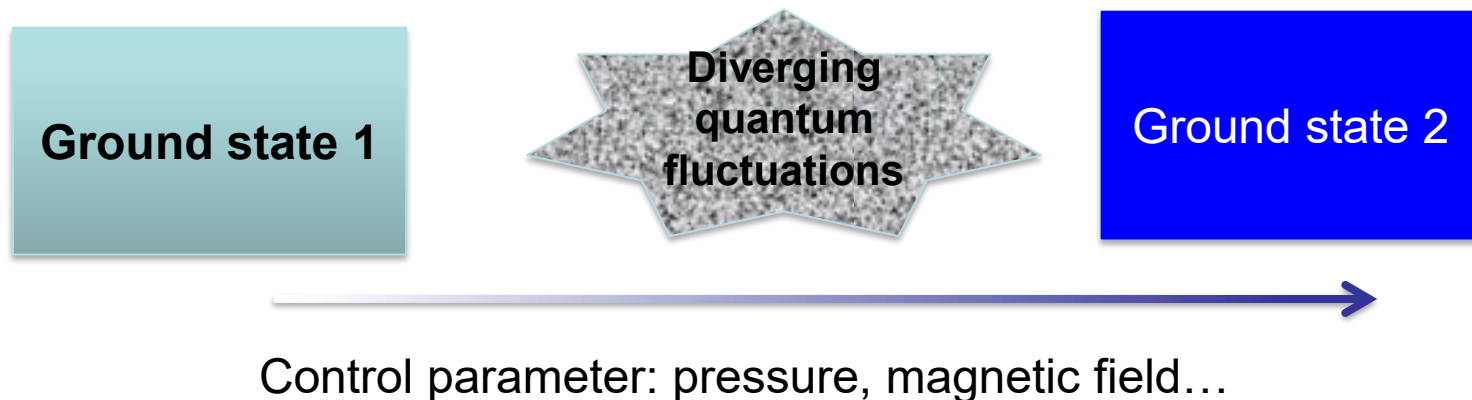
Broken symmetry in the phase diagram full of CPT&QPT



Classical phase transitions



Quantum phase transitions



Index for phase transitions

- How to describe phase transitions? Hamiltonian & effective theories. Symmetries
- Classical phase transitions
 - Landau theory and beyond
 - Microscopic theory. Example: Ising model
 - Micro-Macro bridge: Ising $\leftrightarrow \Phi^4$ theory
 - Criticality
 - Important theorems for continuous symmetry: Goldstone theorem, Mermin-Wagner Theorem, Kosterlitz-Thouless transition.
- Quantum phase transitions QPT
 - Quantum Ising model $\leftrightarrow$ Corresponding effective theory
 - Mapping QPT-CPT
- Comparison CPT & QPT
- Outlook

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Hamiltonian description (micro scale, high energy, lattice...)

$$H(x_1, \dots, x_N) \psi_1(x_1) = E \psi_1(x_1)$$

.

.

.

$$H(x_1, \dots, x_N) \psi_N(x_1) = E \psi_N(x_1)$$

$$H(x_1, \dots, x_N) = H_{electronic}(x_1) + H_{elec,ion}(x_1, \dots, x_N) + H_{elec,elec}(x_1, \dots, x_N) + H_{ion}(x_1, \dots, x_N)$$

In principle we get ground state and excitations

Too many degrees of freedom! Even more for phase transitions

Simplification: Ising model (Spin)..., Hubbard model (spin and charge)... still complicated, even more for phase transitions

Effective theory (macro scale, low energy, continuum)

In the effective theory there are just the relevant degrees of freedom Φ close to the phase transition

Example for effective action

$$\mathcal{S}[\Phi] = \int d^d x \left(\frac{1}{2} (\partial \Phi)^2 + \frac{r}{2} \Phi^2 + g \Phi^4 \right)$$

How to choose the relevant degrees of freedom?

Symmetry

Symmetry in Physics is the invariance of a physical law under transformations.

Symmetries form a group ($O(n)$, $SU(n)$...)

For example, for the group $O(3)$, the rotational group

R is orthogonal if $RR^T = 1$

If $R_1, R_2 \in O(3)$ $R_1 R_2 \in O(3)$

$(R_1 R_2)^T R_1 R_2 = R_2^T R_1^T R_1 R_2 = 1 \rightarrow$ R forms a group

Symmetries in condensed matter

Symmetry	System	Order parameter	Conjugate field	Physical example
Discrete				
Z_2 /Ising	uniaxial FM uniaxial AF	$\langle m_z \rangle$ $\langle m_{z,A} - m_{z,B} \rangle$	h_z $h_{z,A} - h_{z,B}$	Rb ₂ NiF ₄ , K ₂ MnF ₄
Continuous $O_2/U(1)$	order-disorder	$\langle n_A - n_B \rangle$	$\mu_A - \mu_B$	β -brass
	displacive	$\langle u_z \rangle$	f_z	BaTiO ₃
	liquid-gas	$\langle n_L - n_G \rangle$	μ	many
	easy-plane FM	$\langle \mathbf{m} \rangle$	$\mathbf{h}$	
	easy-plane AF	$\langle \mathbf{m}_A - \mathbf{m}_B \rangle$	$\mathbf{h}_A - \mathbf{h}_B$	
Internal: gauge symmetries	superfluid	$\langle \psi \rangle$	h_ψ	He ₄
	smectic-C			Fig. 2.7.1
	hexatic-B	$\langle e^{6i\theta} \rangle$		Fig. 2.7.7
O_3	Heisenberg FM	$\langle \mathbf{m} \rangle$	$\mathbf{h}$	EuS, EuO, Fe, Ni
O_0	Heisenberg AF SAW	$\langle \mathbf{m}_A - \mathbf{m}_B \rangle$	$\mathbf{h}_A - \mathbf{h}_B$	RbMnF ₃ polymer

Chaikin & Lubensky

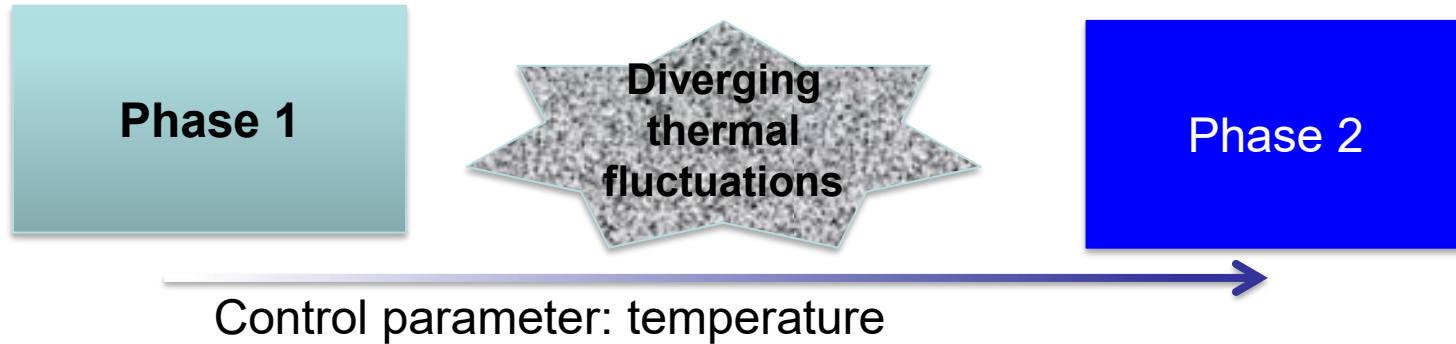
Time-reversal symmetry: broken in ferromagnets, Chern insulator...

And more...can you think of an example?

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Landau theory



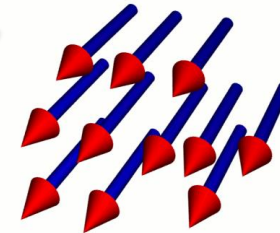
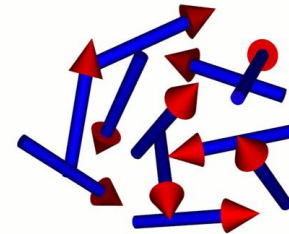
High symmetry (isotropic, homogeneous...),
Kinetic dominates over correlation, higher entropy

At T_c
Symmetry
breaking

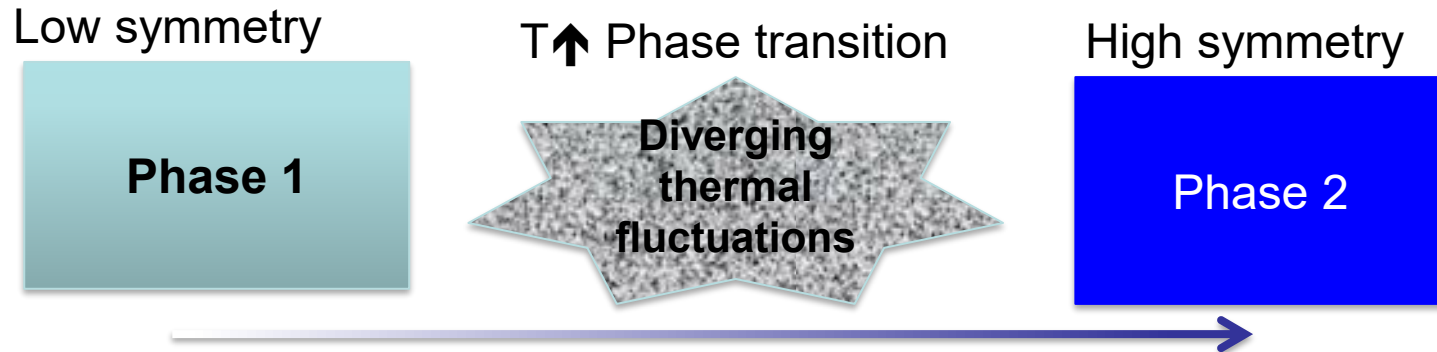
$T \downarrow$

Lower symmetry (order), Correlation
dominates over kinetic energy,
Lower entropy, higher order

Emergence
of FM phase



Landau Theory



Using these facts Landau proposed a theory (assumptions):

1. **Order parameter ϕ** , $\langle \phi \rangle = 0$ in the high symmetry phase and $\langle \phi \rangle \neq 0$ in the ordered phase with lower symmetry. **Symmetry breaking**. Ex magnetization (broken symmetry: spin rotation)
2. Free energy determined minimizing the following functional: $F = F_0(T) + F_L(T, \Phi)$ **Φ controls the thermodynamic.**
3. Building of $F_L(T, \Phi)$: Φ small close to the phase transition, expand $F_L(T, \Phi)$ in powers of Φ respecting symmetries (Ex: $\Phi \rightarrow -\Phi$ discrete broken symmetry)

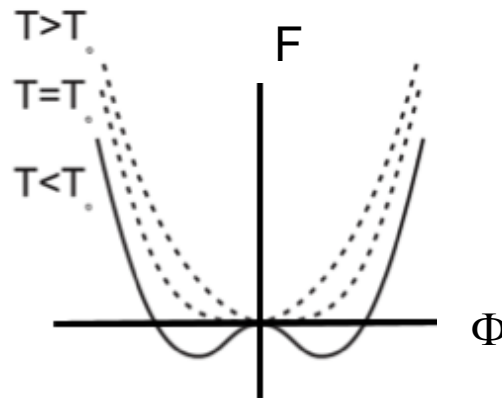
Landau Theory

4. Non-trivial T dependence is in the lowest order signifying the competition between minimizing energy and increasing entropy, $F=E-TS$.

$$F(\Phi, T) = F_0(T) + V \left(\frac{1}{2} r_0 (T - T_c) \Phi^2 + g(T) \Phi^4 \right) \quad \text{Landau Functional}$$

5. Minimizing the Landau functional over Φ we obtain the **mean field theory**.

$$\frac{\partial F}{\partial \Phi} = 0 \rightarrow r_0 (T - T_c) \Phi + 4g(T) \Phi^3 = 0 \rightarrow \bar{\Phi} = \begin{cases} 0 & T > T_c \\ \left(\frac{-r_0 (T - T_c)}{4g} \right)^{1/2} & T < T_c \end{cases}$$



Contact with experiments: susceptibility

For example: magnetization:

$$F[T, m] = \frac{r_0(T - T_c)}{2} m^2 + g(T) m^4 - h m \dots$$

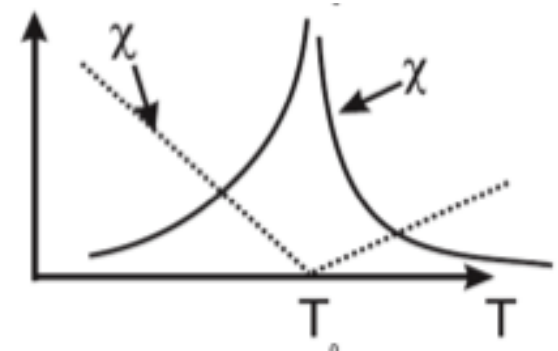
Small (magnetic) perturbation
Linear response

χ Susceptibility:

$$\chi = \partial m / \partial h = \partial^2 F / \partial h^2$$

$$\frac{\partial F[T, m]}{\partial m} = 0 \rightarrow r_0(T - T_c)m + 4g(T)m^3 = h$$

$$\chi_0^{-1} = \frac{\partial h}{\partial m} \propto r_0(T - T_c)$$



It diverges! Second order
phase transition

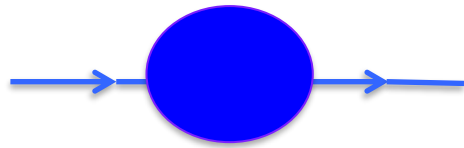
Contact with experiments: Lineal response theory

Small (electromagnetic)
perturbation

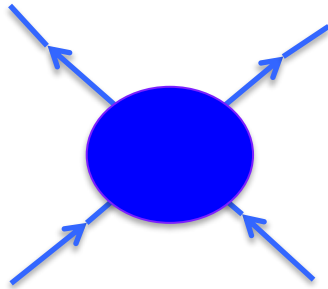


Lineal response

(Ex. $m = \chi H$)



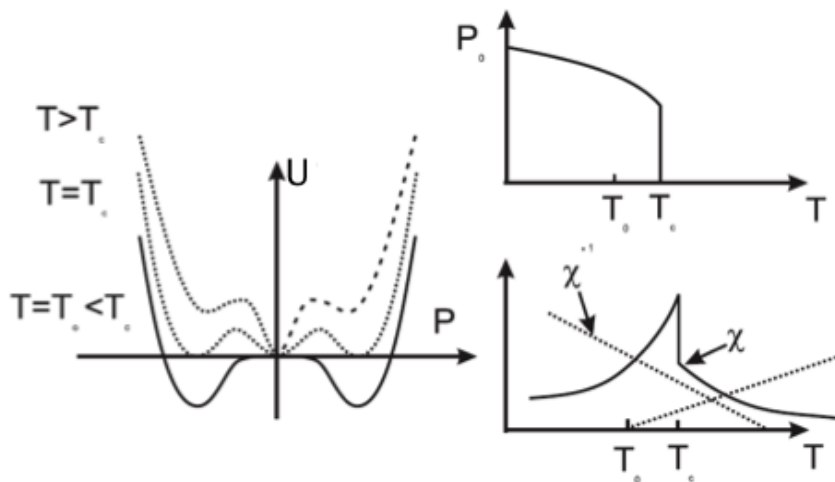
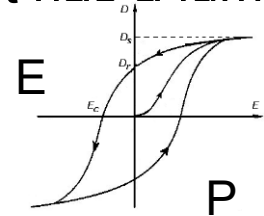
Response to one particle: thermodynamic quantities (specific heat, compressibility, magnetic susceptibility...), ARPES, STM,



Response to two particles: transport (electric current, spin current, Hall current->Kubo formalism), spectroscopic techniques (Raman, X-ray, optical conductivity, neutron scattering...)

1st Order phase transitions

- The order parameter is not a continuous function of T but it has a jump. Susceptibility also with a jump.
- There can be phase separation (negative compressibility) and may exhibit hysteresis.
- Described by a Landau functional with more terms: cubic, sixth with negative fourth...



$$\alpha = \beta(T - T_0)$$

1st Order ($\gamma < 0$)

$$U = \frac{\alpha}{2} P^2 + \frac{\gamma}{4} P^4 + \frac{\delta}{6} P^6 - EP$$

Ginzburg-Landau

Fluctuations around the mean field solution-> Getting closer to Criticality

Long-wavelengths fluctuations can be added in Landau theory fluctuations: $\phi(x)$ is slowly varying and one just keeps the gradient term

$$F_{GL}[T, \Phi] = \int dx^d \left(\frac{1}{2} c (\nabla \Phi)^2 + \frac{r}{2} \Phi^2 + g(T) \Phi^4 - h \Phi \dots \right) \rightarrow \text{Ginzburg-Landau Functional: include fluctuations}$$



$\Phi(x) \rightarrow$ **local** order parameter

A scale is introduced: ξ
Correlation length at $T > T_c$

$$\xi^2 = \frac{c}{r_0(T - T_c)} \rightarrow \xi \text{ Diverges at } T \rightarrow T_c \text{ as } 1/2 \text{ in mean field}$$

For second order phase transitions the correlation length diverges as a power law at the critical point. At the transition point is infinite, implying that the thermodynamic properties C_v , χ , ... also diverge

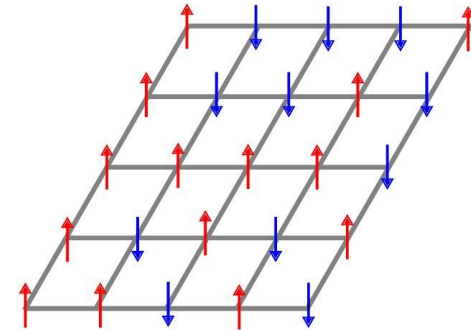
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Spontaneous symmetry breaking in the Ising model

Ex: Consider the Ising model ($S=\pm 1$)

$$H = -J \sum_{\langle ij \rangle} S_i S_j \quad Z = \sum_{\{S_i\}} e^{-\beta H}$$



The hamiltonian possess a discrete symmetry $\rightarrow Z_2$ invariant ($+1 \rightarrow -1$) but in the ordered phase the ground state is not invariant $\rightarrow$ Spontaneous symmetry breaking!

$$\langle m \rangle = \mu \sum_i \langle S_i \rangle = \frac{1}{Z} \sum_i \sum_{\{S_i\}} S_i e^{-\beta H} = 0!!!!$$

In fact it is impossible to observe a phase transition in a finite system

Solution $\rightarrow$ **Thermodynamic limit & extra field (h) in the Hamiltonian**

$$H = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i \quad \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \langle m \rangle = \bar{m} \neq 0$$

Emergence of the FM phase

Limits cannot be interchanged!

Definitions: Correlation function

In Statistical Physics all the information is encoded in the partition function
 The probability of finding the system in the microstate i with energy E_i is given in thermal equilibrium by the Boltzmann factor: $\exp(-E_i/k_B T)/Z$

$$Z = \sum_{\{S\}} e^{-\beta H} \rightarrow \text{Generating function} \quad Z = e^{-\beta F} \rightarrow F = -\frac{1}{\beta} \ln Z$$

$$H = -J \sum_{\langle ij \rangle} S_i S_j - \mu \sum_i B_i S_i \quad B_i \rightarrow \text{fictitious (conjugate) field}$$

$$\langle S_i \rangle = \frac{1}{Z} \sum_{\{S\}} e^{-\beta H} S_i = \frac{1}{(\beta \mu)} \frac{\partial \ln Z}{\partial B_i} \bigg|_{B=0} \quad m = \mu \sum_i \langle S_i \rangle$$

$$G_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = \frac{1}{(\beta \mu)^2} \frac{\partial^2 \ln Z}{\partial B_i \partial B_j} \bigg|_{B=0}$$

Correlation function: measure how correlated is the system over a certain range of space

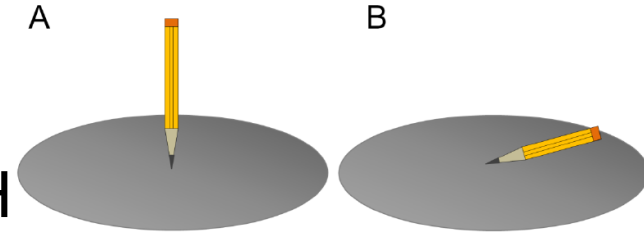
$$\chi = \frac{\partial m}{\partial B} = \frac{1}{\beta} \sum_{ij} \frac{\partial^2 \ln Z}{\partial B_j \partial B_i} = \beta \mu^2 \sum_{ij} \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

Fluctuation-dissipation theorem:

$\chi \rightarrow$ response function
 Correlations can be measure!

**At a phase transition χ diverges -
 > infinite fluctuations**

In general: Spontaneous Symmetry Breaking SSB



G a symmetry group of the Hamiltonian H
 Then SSB occurs if in the ground state ($T=0$) at the thermodynamic limit

$$\lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \langle m \rangle = \bar{m} \neq 0$$

When $|i-j| \rightarrow \infty$ (long wavelengths),

$$G_{ij} = \bar{m}^2 \rightarrow \text{long-range ordered}$$

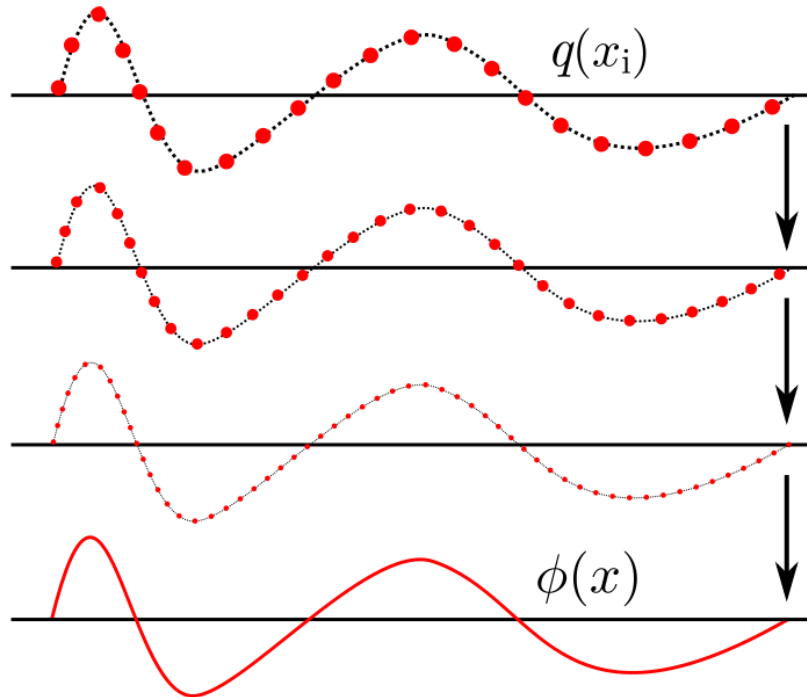
observable not G -inv. $\rightarrow$ The ground state is not invariant under G

$\bar{m}$: order parameter (of the new phase)

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Definition: Continuum limit



$$N \rightarrow \infty, V \rightarrow \infty, N/V \rightarrow \text{finite}$$

$$x_i = a i \rightarrow x$$

$$\sum_{i=1}^N \rightarrow \frac{1}{a} \int_0^L dx$$

$$\frac{\phi_{i+1} - \phi_i}{a} \rightarrow \frac{\partial \phi}{\partial x}$$

$\Phi(x)$ is a smooth function (**field**): relative fluctuations on atomic scales are weak $\rightarrow$ **Field theory**

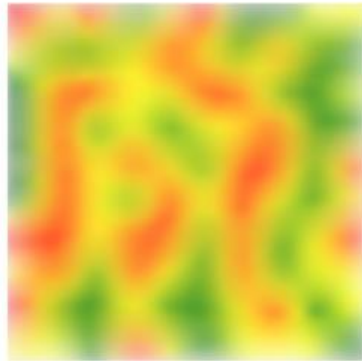
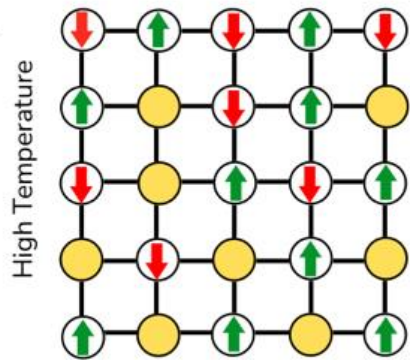
Macro-Micro bridge

Microscopic Model

Each site is a vital norm, the spin configuration is the internal evaluation of the norm

Continuos Model

Local perception $\phi(x)$

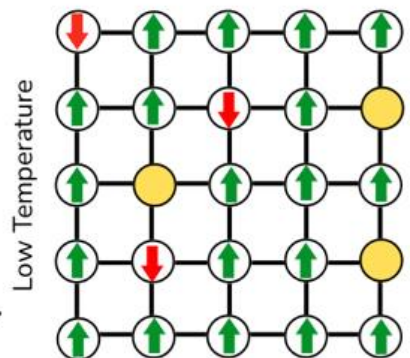


Emergence of macroscopic patterns from microscopic degrees of freedom:

Intuition:

$$Z = e^{-\beta F} \approx e^{-\beta F_0} e^{-\min_{\{\Phi\}} \beta F_L[T, \Phi]}$$

Microstate



Formal theory

Saddle point approach:

$$Z = e^{-\beta F} = \sum_{\{\mu\}} e^{-\beta H[\mu]} \approx e^{-\beta F_0} \int D\Phi e^{-F_{GL}[T, \Phi]/k_B T}$$

Formal connection with Landau Theory:

It can be shown that after a *Hubbard-Stratonovich* transformation the Ising model close to the phase transition transforms to the ϕ^4 theory ($\Phi \ll 1$) in the continuum limit

Ising model (discrete)

$$H = -J \sum_{\langle ij \rangle} S_i S_j$$

$$Z = \sum_{\{S_i\}} e^{-\beta H}$$

Close to T_c

Hubb-Strat
Transform.

Landau-Ginzburg functional or
effective action (continuum)

$$\mathcal{S}[\Phi] = \int dx^d \left(\frac{1}{2} (\partial \Phi)^2 + \frac{r}{2} \Phi^2 + g \Phi^4 \right)$$

$$Z = \int D\Phi e^{-\mathcal{S}[\Phi]}$$

For 1D: $r \propto (1 - 2\beta J)$ $g \propto J^2$ Macro parameters in function of micro!

r will change sign at $T_c \rightarrow k_B T_c = 2J$ The same result is obtained using mean field
 $k T_c = q J$ with $q \rightarrow$ number of nearest neighbors. Mean field is equivalent to the
 saddle point approach:

$$\frac{\delta \mathcal{S}[\Phi]}{\delta \Phi} = 0 \rightarrow \Phi_{MF}$$

To study fluctuations one must go beyond the saddle point approximation

Book: Atland & Simons

Fluctuations around mean field: approaching criticality

- ✓ Thermal fluctuations will tend to decrease T_c
- ✓ Validity of mean field: The amplitude of the fluctuations is small if the dimension of the system is larger than a critical dimension ($d_c=4$ for the Ising Universality class). This is stated by the Ginzburg criterion.

$$\frac{\Delta m^2}{m^2} = \frac{g}{6} r_0^{D/2-2} (T_c - T)^{D/2-2}$$

- ✓ Beyond mean field → renormalization group (a procedure to eliminate high energy states keeping only what affects the low energy physics).

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Criticality

Criticality can be studied in the continuum limit: the correlation length is the only relevant length scale:

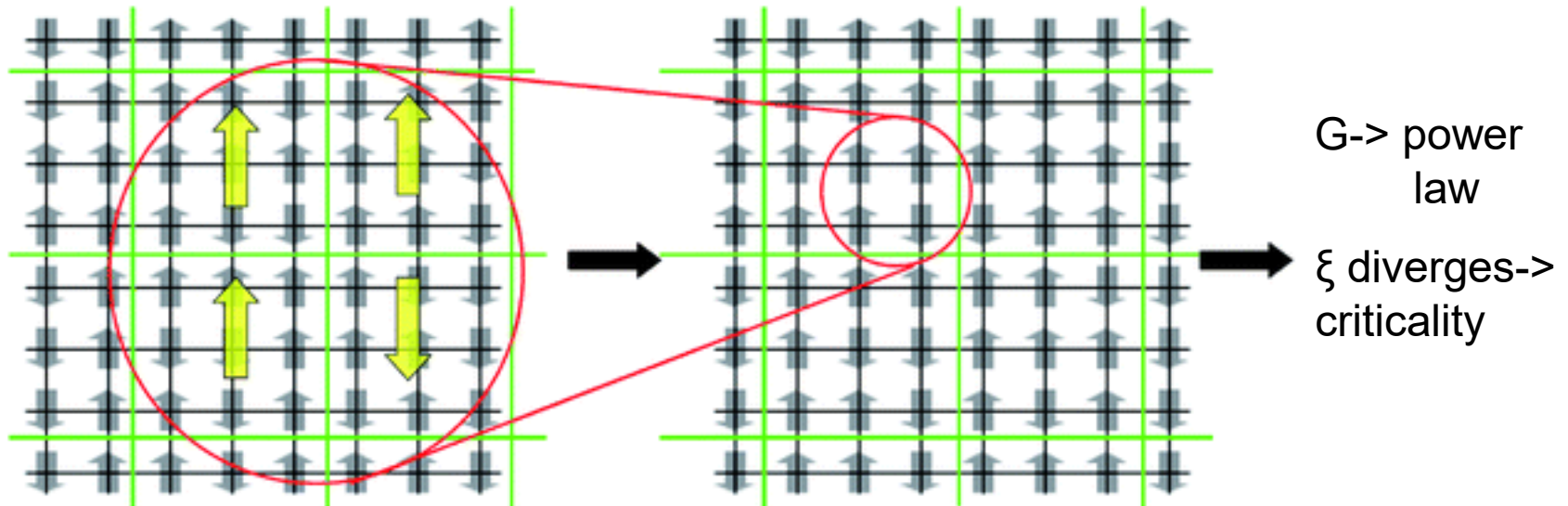
$$G(x, 0) \approx e^{-x/\xi} \quad x \rightarrow \infty \quad \text{Disorder phase} \\ \text{(Proper definition of } \xi \text{)}$$

In the ordered phase:

In the disordered phase at long distances the correlation function:

$$\lim_{x \rightarrow \infty} G(x, 0) = m^2 \quad \text{Ordered phase: Long range order}$$

There must be a phase transition showing critical behavior.



Critical exponents

Experimentally, with scaling arguments, or with renormalization group it can be shown power law decay:

At the critical point
the correlation function $\rightarrow G(x) \sim \frac{1}{x^{D-2+\eta}} \quad G(k) \sim \frac{1}{k^{2-\eta}}$

Correlation length $\rightarrow \xi \equiv 1/r^\nu \quad r = r_0(T - T_c)$

Using the fluctuation-dissipation theorem

$$\chi \approx |r|^{-\gamma} = |r|^{-\nu(2-\eta)} \quad \rightarrow \text{experimentally}$$

γ, ν and η ... are critical exponents: set of different exponents characterizing the transition. They are *universal*.

Typical critical exponents:

Order parameter:	$\psi \propto \begin{cases} (T_c - T)^\beta \\ h^{1/\delta} \end{cases}$	Note that the Landau theory for an Ising model (or ϕ^4 theory) gives
Correlation length:	$\xi \propto (T_c - T)^{-\nu}$	
Susceptibility:	$\chi \propto (T_c - T)^{-\gamma}$	
Specific heat:	$C \propto (T_c - T)^{-\alpha}$	
Correlation function:	$G_c^{(2)}(r) \sim \frac{1}{r^{d-2+\eta}}$ large and $T=T_c$	
		$\beta=1/2$ $\nu=1/2$ $\gamma=1$ $\alpha=0$ $\delta=3$

The exponents are related:
 (only 2 are independent)

$$2 - \alpha = 2\beta + \gamma$$

$$(2 - \eta)\nu = \gamma$$

$$2 - \alpha = \beta(\delta + 1)$$

$$\nu d = 2 - \alpha$$

Universality

At $T_c \rightarrow$ criticality, scale invariance, cooperative phenomena $\rightarrow$ the properties (critical exponents) depend on

- Dimensionality of the space
- Dimensionality of the order parameter
- Symmetries of the local couplings

But not on the details of the interaction.

Ex: For the Ising model the square and the triangular lattice have the same critical exponents (at T_c the system is rotational invariance)

Ex: the solid and liquid or solid and gas transition (water, carbon dioxide...) belongs to the same universality class than the magnetic transition

Magnetic transition	$\longrightarrow$	Solid liquid/gas transition
Fluctuation magnetization	$\Rightarrow$	fluctuating density
Order parameter m	$\longrightarrow$	density
Neutron scattering	$\longrightarrow$	light scattering
$M \propto (T_c - T)^\beta$		$ \rho_L - \rho_g \propto T_c - T ^\beta$

Better understood in the RG framework (fixed point)

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Continuous symmetry $\Phi = (\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x))$

$$\Phi^2 \rightarrow |\Phi|^2 = \sum_{i=1}^N \Phi_i^2$$

$$(\nabla \Phi)^2 \rightarrow |\nabla \Phi|^2 = \sum_{i=1}^N (\partial_x \Phi_i)^2$$

$$h\Phi \rightarrow h \cdot \Phi \quad \text{Conjugate field}$$

Important theorems:

- Goldstone theorem
- Mermin-Wagner theorem
- 2D $\rightarrow$ Kosterlitz-Thouless transition

Spontaneous symmetry breaking of a continuous symmetry: Goldstone Theorem

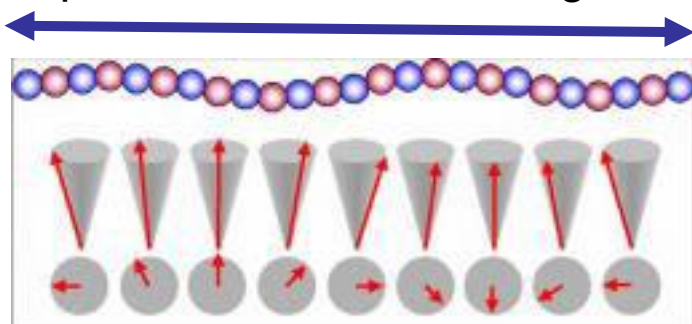
Low energy (long wavelength) excitations are possible in systems with continuous symmetry. The excitations are Goldstone Modes -> EMERGENT QUASIPARTICLES

Example: Heisenberg model with S

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle} \cos \theta_{ij}, \quad \vec{S}_i \in \mathbb{R}^3, \quad |\vec{S}_i| = 1$$

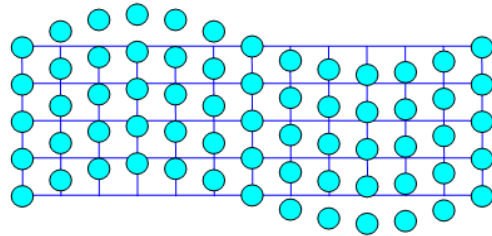
$E \sim J(1 - \cos \theta) \sim J\theta^2$ (for small θ) The cost of this energy can be vanishingly small

Spin waves with wavelength λ : magnons



Other goldstone modes

✓ In a crystalline solid: acoustic phonons
 $E \sim |q|$.

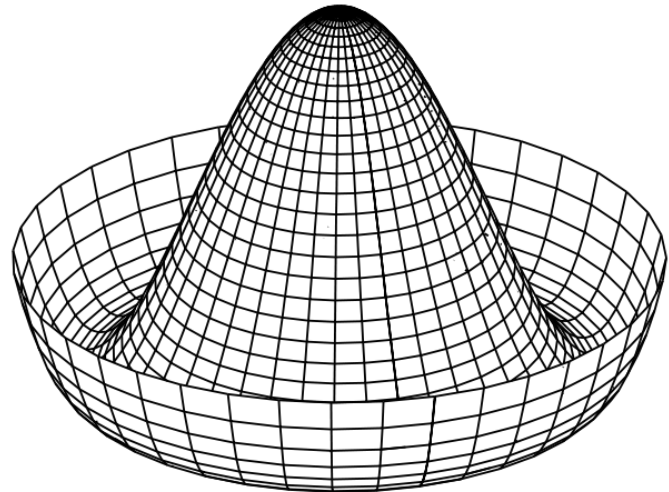


✓ In a superfluid (neutral fluid), Bogoliubov modes.

ϕ^4 theory, Φ complex U(1) symmetry

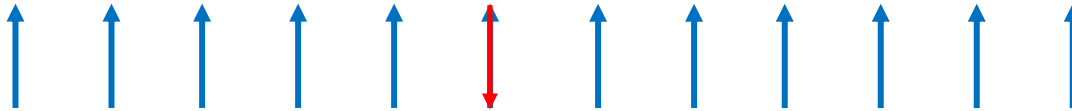
$$F = (\partial_\mu \phi)(\partial^\mu \phi^*) + m^2 \phi^* \phi + \lambda (\phi^* \phi)^2$$

$$\phi \rightarrow e^{i\Lambda} \phi \quad -V(\phi)$$



No Goldstone modes

The Ising model has discrete symmetry:
 $E \sim J$ (domain wall) every excitation costs finite energy



A charged fluid (superconductor) develops a gapped spectrum:
Anderson-Higgs mechanism due to the electromagnetic field.

Mermin-Wagner theorem

(Phase transitions and dimensionality)

Mermin, N. D. & Wagner, H. *Phys. Rev. Lett.* **17**, 1133–1136 (1966).

Goldstone modes gives rise to large fluctuation effects in low dimensions

In general if we have:

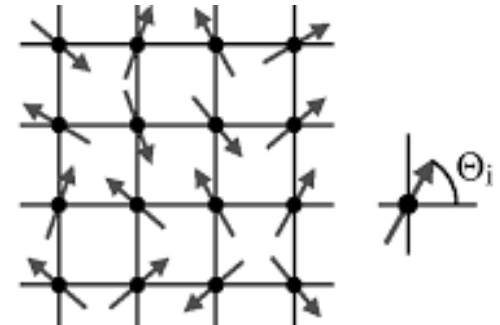
- Spontaneous symmetry of a continuous group (i.e. not applicable to Ising model)
- Short range forces

Then there is no phase transition (associated with a long range order!) for dimension $d \leq 2$ (for $T > 0$).

The special case of $d=2$ and $D=2$: the Kosterlitz-Thouless transition

The XY model ($D=2$) in 2dim ($d=2$) has continuous symmetry $U(1)/O(2)$, hence it cannot have a phase transition to a long-range ordered state (Mermin-Wagner).

$$\mathbf{S}_i = (\cos \theta_i, \sin \theta_i)$$
$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$



Berezinskii (1970), Kosterlitz and Thouless (1972) demonstrated that the system undergoes a phase transition (though not long range ordered, there is not symmetry breaking).

The special case of $d=2$ and $D=2$: the Kosterlitz-Thouless transition

Solution: vortices (topological defects) that contribute to the entropy.

Low T: vortex-antivortex binding High T: Unbinding vortices

Quasi long range order

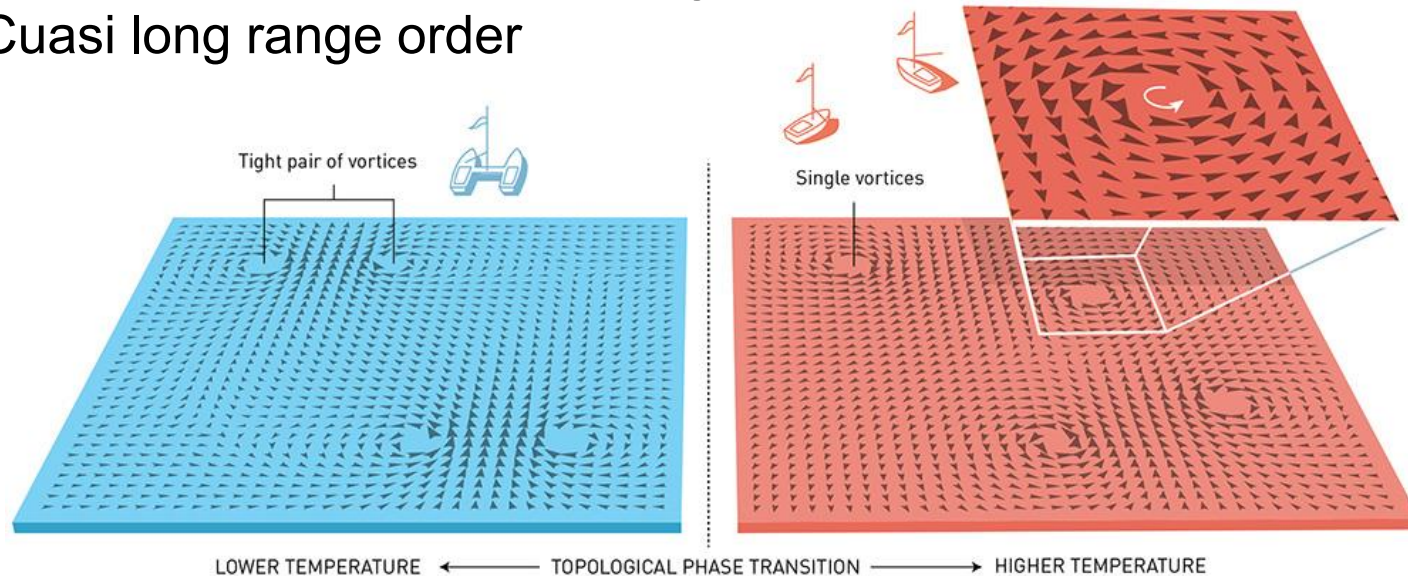


Illustration: ©Johan Jarnestad/The Royal Swedish Academy of Sciences

Algebraic
decay!

$$G(x,0) \approx \left(\frac{a}{|x|} \right)^{\frac{b}{\beta}} \quad x \rightarrow \infty$$

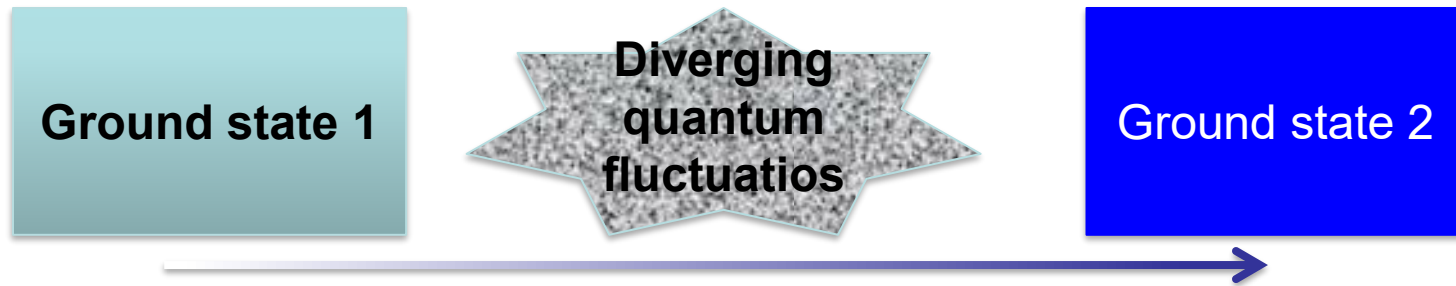
$$G(x,0) \approx e^{-x/\xi} \quad x \rightarrow \infty$$

XY universality class: Topological defects (Coulomb charges in 2D, dislocations in 2D crystals, vortices in 2D superconductors ($\Psi \sim e^{iq}$) or superfluids...)

Quantum phase transitions

Quantum phase transitions

QPT: $T=0$

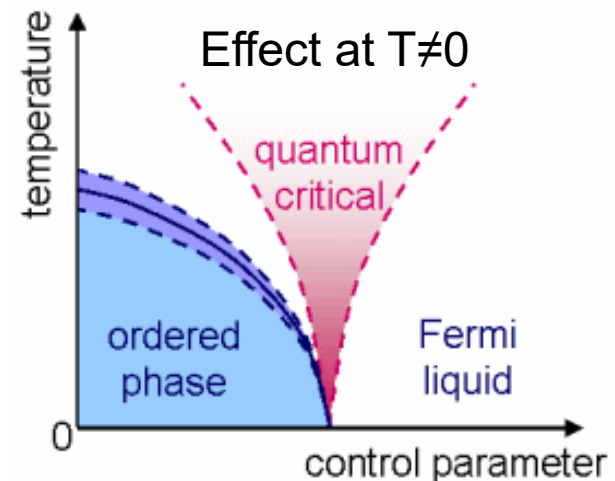


Control parameter: pressure, magnetic field...

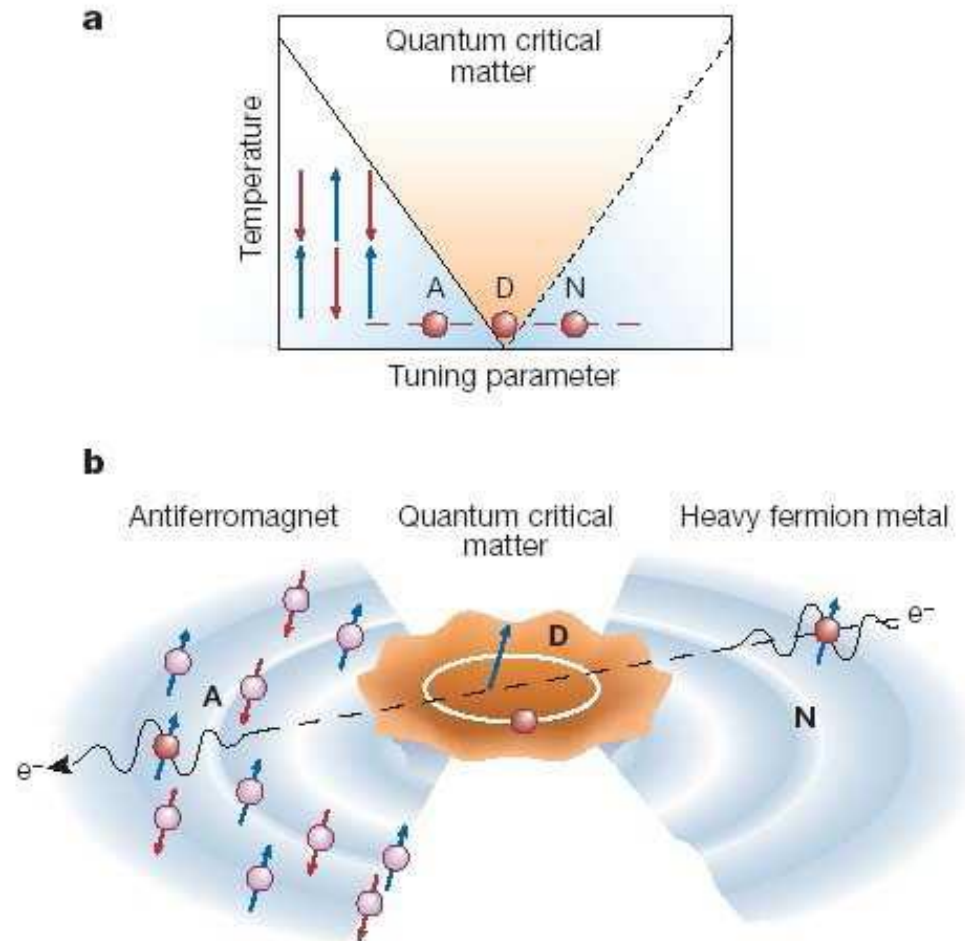
Quantum fluctuations are driven by the Heisenberg uncertainty principle

$$H(g) = H_0 + gH_1$$

- Transition at g_c (a point of non-analyticity of the ground state).
- The nature of the correlations in the ground state changes qualitatively at g_c .

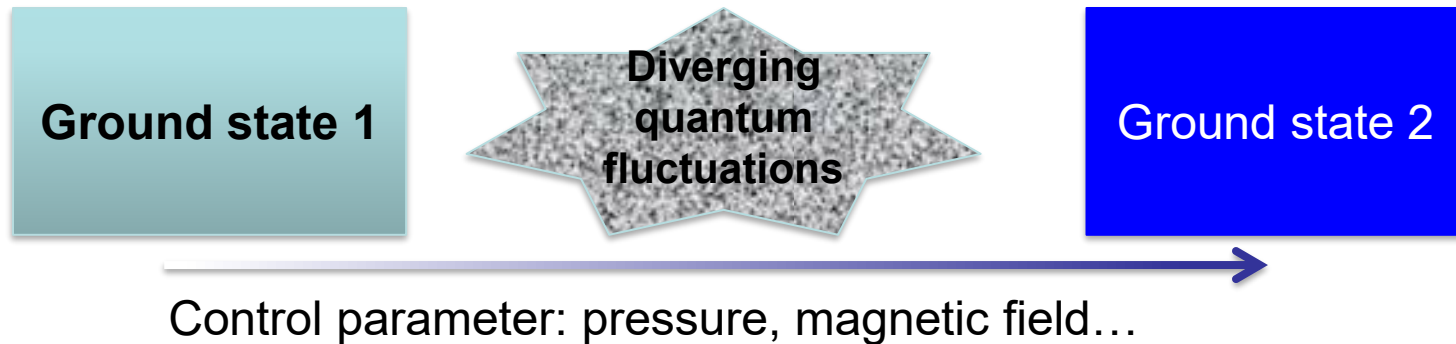


Non Fermi liquid behavior



Quantum phase transitions

QPT: $T=0$



QPT have consequences at $T \neq 0$.

$$\xi^{-1} \sim |g - g_c|^{\nu}$$

$$\tau_{\varphi} \sim \xi^z = |g - g_c|^{-\nu z}$$

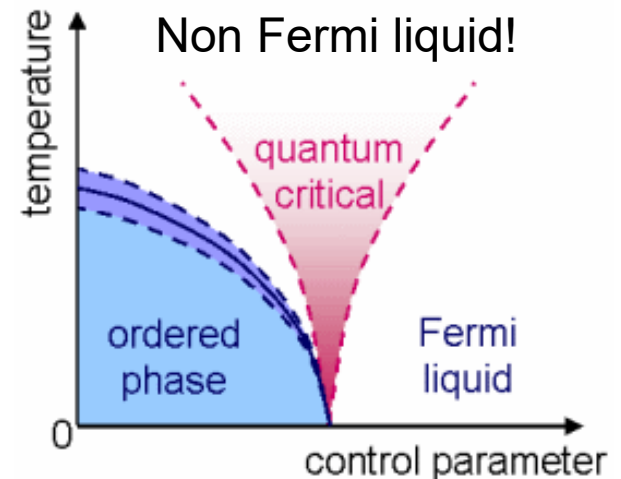
$$\Delta \sim |g - g_c|^{\nu z}$$

} New! $|\varphi, z$

z is the dynamical exponent, Δ is the gap

$|\varphi$ is the coherence time, time over which the wave function retains its memory of its phase: it diverges at a QPT

2nd order QPT:



Example: Quantum Ising model

Ising model in a transverse field

$$H = -Jg \sum_i \hat{\sigma}_i^x - J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

$$\hat{\sigma}_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_i^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For $g=0 \rightarrow$ Ising model $\sigma_i^z=1,-1$ $|\uparrow\rangle, |\downarrow\rangle \rightarrow$ *Ordered phase*

For $g \neq 0$ $\sigma_i^x \rightarrow$ off diagonal $\rightarrow$ quantum mechanical tunneling $|\uparrow\rangle \rightarrow |\downarrow\rangle$
 $\rightarrow$ *Disordered phase*

This model can describe real systems such as LiHoF_4 where it has been identified a QCP

Example: Quantum Ising model $T=0$, $g \gg 1$ limit

Ising model in a transverse field

$$H = -Jg \sum_i \hat{\sigma}_i^x - J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

Limits:

$g \gg 1$ 1st term dominates: quantum paramagnetic (QPM)

$$|0\rangle = \prod_i |\rightarrow\rangle_i$$

$$|\rightarrow\rangle_i = (|\uparrow\rangle_i + |\downarrow\rangle_i) / \sqrt{2} \quad (-1)$$

$$|\leftarrow\rangle_i = (|\uparrow\rangle_i - |\downarrow\rangle_i) / \sqrt{2} \quad (1)$$

$$|i\rangle = |\leftarrow\rangle_i \prod_{j \neq i} |\rightarrow\rangle_j$$

The system is totally uncorrelated. 1st excitations (gap= Δ)

$$\langle 0 | \hat{\sigma}_i^z \hat{\sigma}_j^z | 0 \rangle \propto e^{-|x_i - x_j|/\xi}$$

Example: Quantum Ising model $T=0$, $g \ll 1$ limit

Ising model in a transverse field

$$H = -Jg \sum_i \hat{\sigma}_i^x - J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

Limits:

$g \ll 1$ 2nd term dominates: Z_2 symmetry: Magnetic long range order (MLRO)

$$|\uparrow\rangle = \prod_i |\uparrow\rangle_i \quad \text{or} \quad |\downarrow\rangle = \prod_i |\downarrow\rangle_i$$

Excitations (gap= Δ): domain walls: turning on g will mix up and down spins but still Z_2 symmetry.

$$|-\rangle |-\rangle | \rangle | \rangle | \rangle | \rangle |-\rangle |-\rangle$$

Only in the thermodynamic limit:

$$\lim_{|x_i - x_j| \rightarrow \infty} \langle 0 | \hat{\sigma}_i^z \hat{\sigma}_j^z | 0 \rangle \approx M^2 \quad M=1 \text{ if } g=0$$

Example: Quantum Ising model

T=0, Quantum phase transition

Ising model in a transverse field

$$H = -Jg \sum_i \hat{\sigma}_i^x - J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

Limits:

$$\left. \begin{array}{l} g \gg 1 \quad \langle 0 | \hat{\sigma}_i^z \hat{\sigma}_j^z | 0 \rangle \approx e^{-|x_i - x_j|/\xi} \\ g \ll 1 \quad \lim_{|x_i - x_j| \rightarrow \infty} \langle 0 | \hat{\sigma}_i^z \hat{\sigma}_j^z | 0 \rangle \approx M^2 \end{array} \right\} \text{At } g=g_c \text{ we have a QPT, we expect } \xi \text{ diverges and } G \text{ goes as a power law}$$

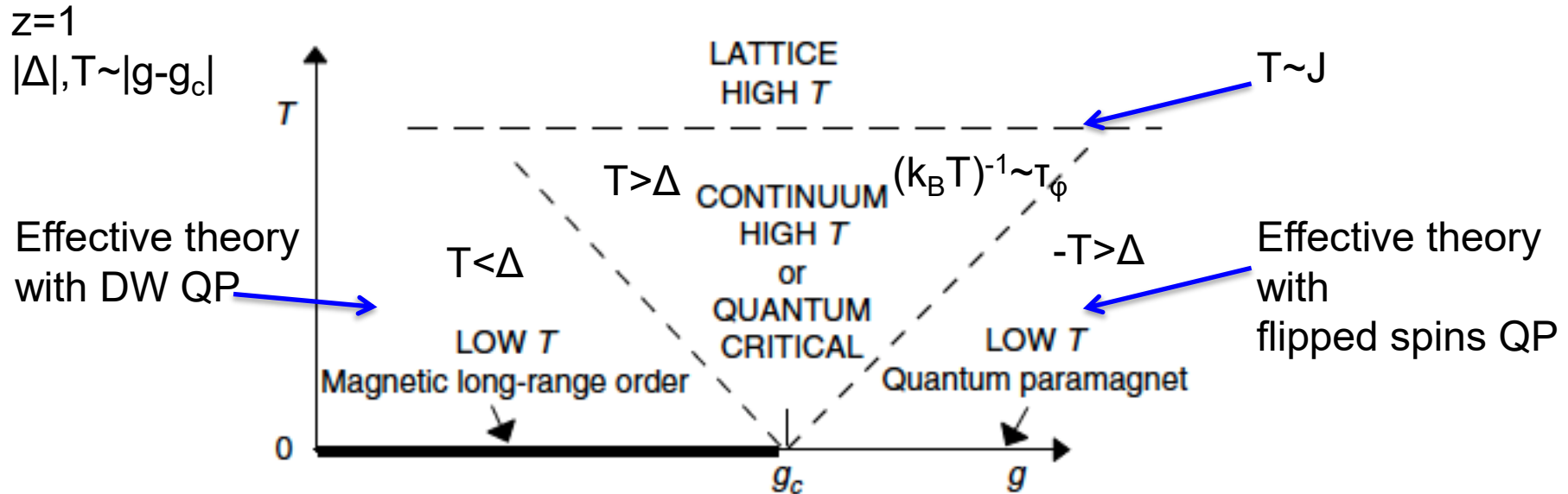
It is possible to calculate the exact spectrum using a Jordan-Wigner transformation (mapping between spin $\frac{1}{2}$ degrees of freedom and spinless fermions). The result is:

$$\varepsilon(k) = 2J(1 + g^2 - 2g \cos k)^{1/2} \quad \Delta = 2J|1-g| \rightarrow g=1 \text{ QCP!!} \quad \left\{ \begin{array}{l} g > 1 \text{ exp. law} \\ g < 1 \text{ long range} \end{array} \right.$$

$$\varepsilon(k \rightarrow 0, g=1) = 2J(2 - 2(1 + \frac{1}{2}k^2))^{1/2} = 2J|k| \rightarrow \text{Excitations without gap!}$$

Example: Quantum Ising model

Phase diagram: Exact solution



Low T (magnetically ordered) quasi-classical particles –domain walls	T Continuum high (quantum critical)	Low T (quantum paramagnetic), quasi-classical particles –flipped spins
$\xi \rightarrow \left(\frac{\pi c^2}{2\Delta T} \right)^{1/2} e^{\Delta/T}$ $\tau_\phi \rightarrow \frac{\pi}{2T} e^{\Delta/T}$	$\frac{4c}{\pi T}$ $\frac{\cot(\pi/16)}{2T}$	$\frac{c}{ \Delta }$ $\frac{\pi}{2T} e^{ \Delta /T}$

relaxational

Sachdev: Quantum phase transitions

Quantum classical mapping

$$Z = \sum_{\{S_i\}} e^{-\beta H} \quad H = H_{kin} + H_{pot}$$

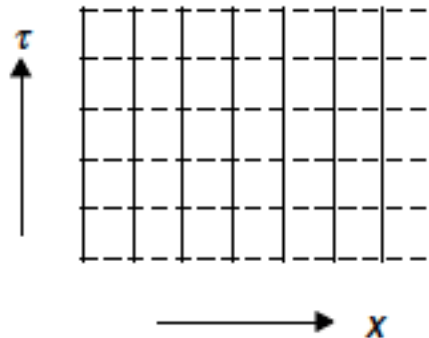
In a classical system $[H_{kin}, H_{pot}] = 0$ thus $Z = Z_{kin} Z_{pot}$ and statics and dynamics are decoupled $\rightarrow$ Effective time-independent theories in D dimensions.

In a quantum system $[H_{kin}, H_{pot}] \neq 0$ thus **statics and dynamics are coupled**. $\phi(x, \tau)$. The operator $e^{-H/kT}$ looks like $e^{-H\tau}$

$1/kT = \tau = -it/\hbar \rightarrow$ Mapping

D (>1 dimensional classical) = d (dimensional quantum) + z

These models belong to the same universality class



z measures the anisotropy between x-dim and t-dim

Are QPT($d+z$) different from CPT(D)?

- The Quantum-Classical mapping yields quantum correlation functions that are in imaginary time. The analytical continuation to real time is highly non-trivial.
- It emerges a new time scale: the coherence time τ_ϕ (not present in the classical analog)
- Berry physics
- Emergence of the quantum critical region

Classical Phase Transitions

- CPT are points or sets of points in the phase diagram which are singularities in the free energy as a function of T. SSB mechanism. 1st and 2nd order.
- Thermal fluctuations arises from the competition between S and E
- Kinetic and Potential energy are decoupled. (z no special role)
- The critical exponents characterize the CPT
- At T_c the correlation length diverges

Quantum Phase Transitions

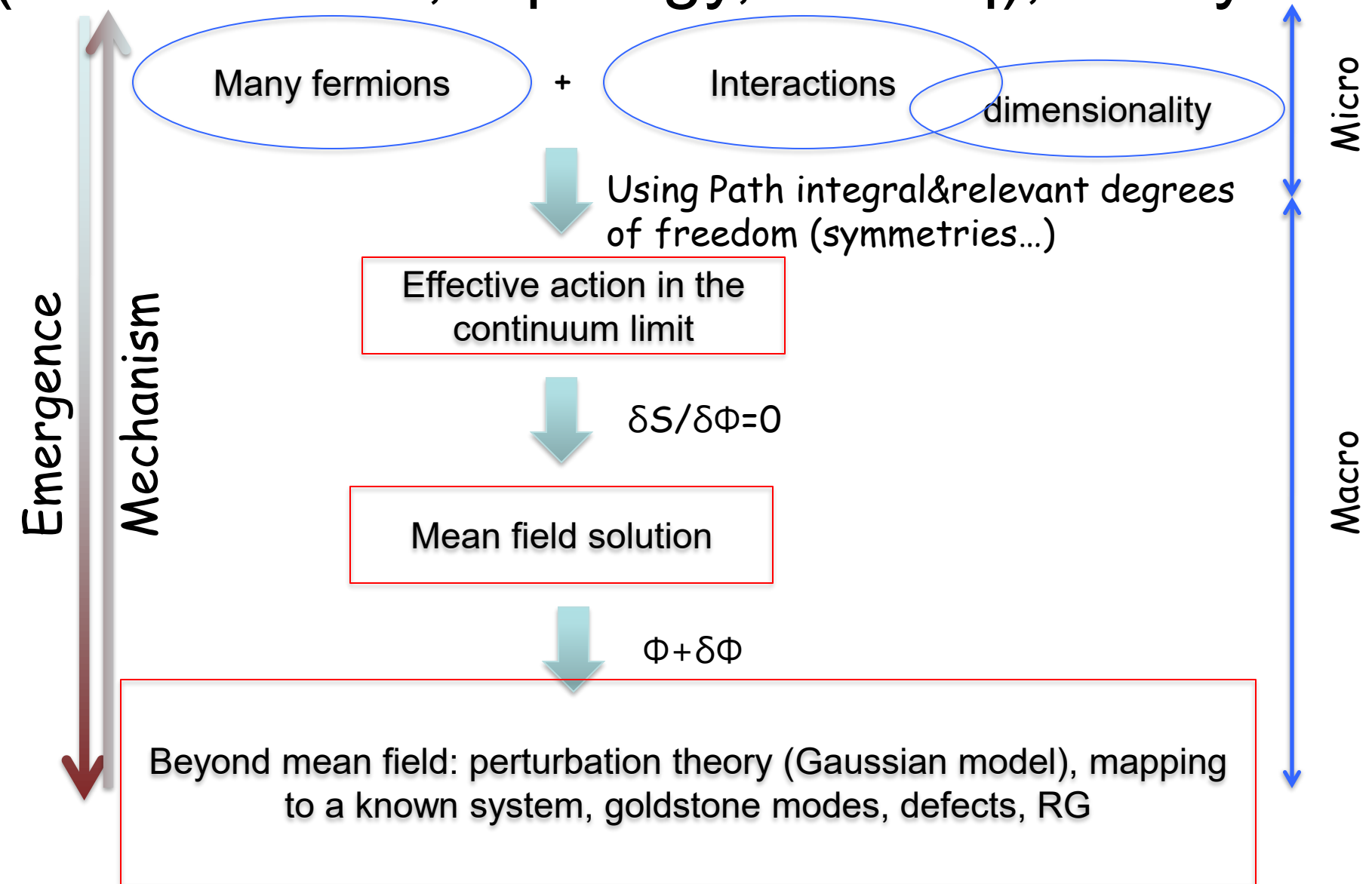
- QPT are points or sets of points non-analytic as a function of a non thermal parameter. SSB mechanism. 1st and 2nd order.
- Quantum fluctuations arises from the competition between different ground states.
- Kinetic and Potential energy are coupled,
- z characterized the transitions together with the other critical exponents.
- At T_c the coherent time and the correlation length diverges
- Quantum critical region (NFL)

Quantum classical mapping:

$D (>1 \text{ dimensional classical}) = d (\text{dimensional quantum}) + z$
(it has its limitations)

Condensed Matter Field Theory

(Hubbard-like, topology, non-eq), life systems



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¡Gracias!

