

Introduction to Topological Superconductivity and Majoranas

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Reviews:

- J. Alicea, Rep. Prog. Phys. 75, 076501(2012).
- Elliot&Franz, Rev. Mod. Phys. 87, 137 (2015).
- R. Aguado, La Rivista del Nuovo Cimento 40, 523 (2017)

Popular article: Ramón Aguado and Leo Kouwenhoven, Physics Today June 2020.



$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0$$



- Since the γ matrices are in general complex, the solutions of Dirac's equation are complex. This is expected since electrons are electrically charged and this requires complex spinors.
- Majorana's question (1937): does the Dirac equation necessarily involve complex solutions?

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0$$

Majorana's answer: No, if Weyl matrices are purely imaginary (Majorana representation)

$$\gamma^0 = \sigma_y \otimes \sigma_x$$

$$\gamma^1 = i\sigma_x \otimes 1$$

$$\gamma^2 = i\sigma_z \otimes 1$$

$$\gamma^3 = i\sigma_y \otimes \sigma_y$$

$$\Psi = \Psi^*$$

- Neutral particle equals its own antiparticle.
- Very relevant in high-energy physics (neutrinos, dark matter, etc).
- Many experimental efforts to search for Majorana neutrinos are underway. To date their detection remains an experimental challenge.

More rigorously, the particle and antiparticle are charge conjugate solutions of the Dirac equation coupled to the electromagnetic field.

$$\begin{aligned}\partial_\mu &\rightarrow i\partial_\mu + eA_\mu \\ \partial_\mu &\rightarrow i\partial_\mu - eA_\mu\end{aligned}$$

RELEVANT FOR US: in the most general case, it is sufficient to demand in any representation that there exists a matrix such that both charge conjugate solutions are equal

$$\Psi^C = C\Psi^* = \Psi$$

$$C = \tau_y \otimes \sigma_y = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$$

charge conjugation operator

Dirac equation in Weyl representation (chiral spinors where the spin of a particle points in the same direction as its momentum)

$$\begin{pmatrix} -m & i\partial_t - \boldsymbol{\sigma}\mathbf{p} \\ i\partial_t + \boldsymbol{\sigma}\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

$$(i\partial_t - \boldsymbol{\sigma}\mathbf{p})\psi_R - m\psi_L = 0$$

$$(i\partial_t + \boldsymbol{\sigma}\mathbf{p})\psi_L - m\psi_R = 0$$



In the Dirac equation, the mass term mixes left and right chiralities

Dirac equation in Weyl representation (chiral spinors where the spin of a particle points in the same direction as its momentum)

$$\begin{pmatrix} -m & i\partial_t - \boldsymbol{\sigma}\mathbf{p} \\ i\partial_t + \boldsymbol{\sigma}\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

Impose charge conjugation $\Psi^C = \Psi$

$$\begin{aligned} (i\partial_t - \boldsymbol{\sigma}\mathbf{p})\psi_R - im\sigma_y\psi_R^* &= 0 \\ (i\partial_t + \boldsymbol{\sigma}\mathbf{p})\psi_L - im\sigma_y\psi_L^* &= 0 \end{aligned}$$

In the Majorana equation, chiralities become decoupled and **the mass term mixes particles and antiparticles** (technically, this implies that the equation is not gauge invariant, hence Majorana particles cannot be coupled to the electromagnetic field and are thus necessarily charge neutral).

perspective

NATURE PHYSICS | VOL 5 | SEPTEMBER 2009 | www.nature.com/naturephysics

Majorana returns

Frank Wilczek

www.sciencemag.org SCIENCE VOL 332 8 APRIL 2011
Published by AAAS

Search for Majorana Fermions Nearing Success at Last?

Researchers think they are on the verge of discovering weird new particles that borrow a trick from superconductors and could give a big boost to quantum computers

Physics

Physics 3, 24 (2010)

Viewpoint

Race for Majorana fermions

Marcel Franz

Department of Physics and Astronomy, University of British Columbia, Vancouver

Published March 15, 2010

The race for realizing Majorana fermions—elusive particles that we still await ideal materials to work with.

Physics

Physics 4, 67 (2011)

Viewpoint

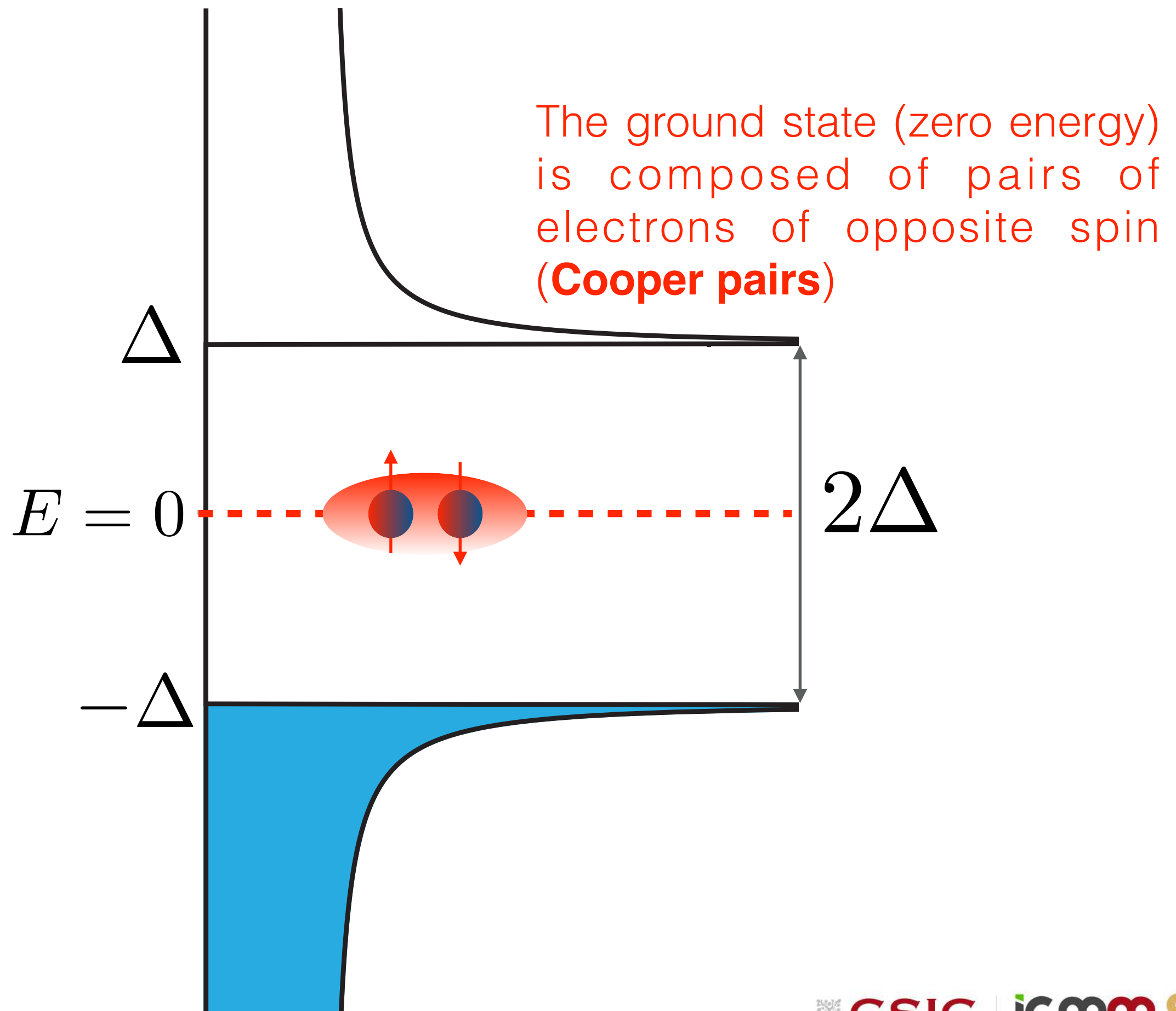
Majorana fermions inch closer to reality

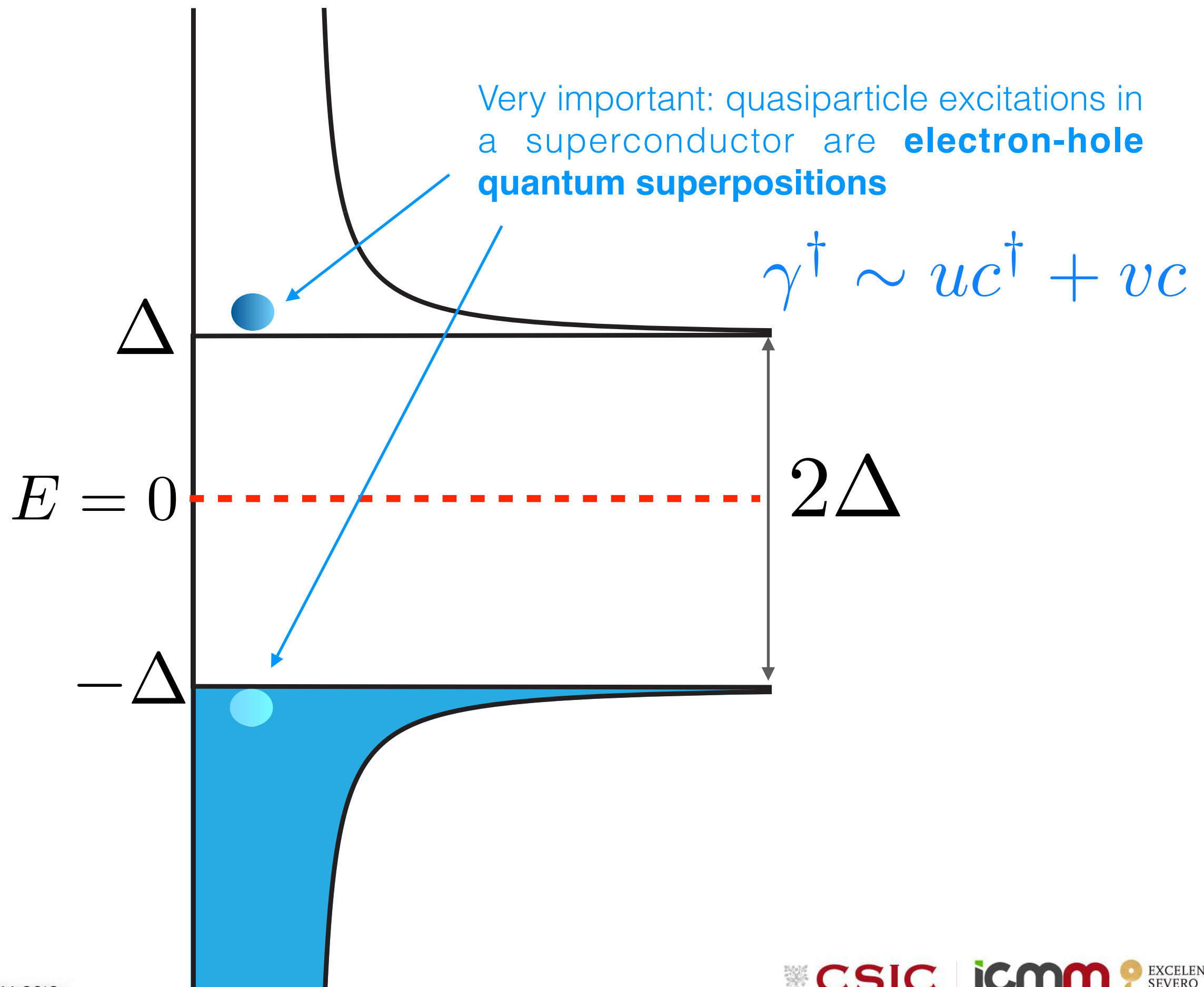
Taylor L. Hughes

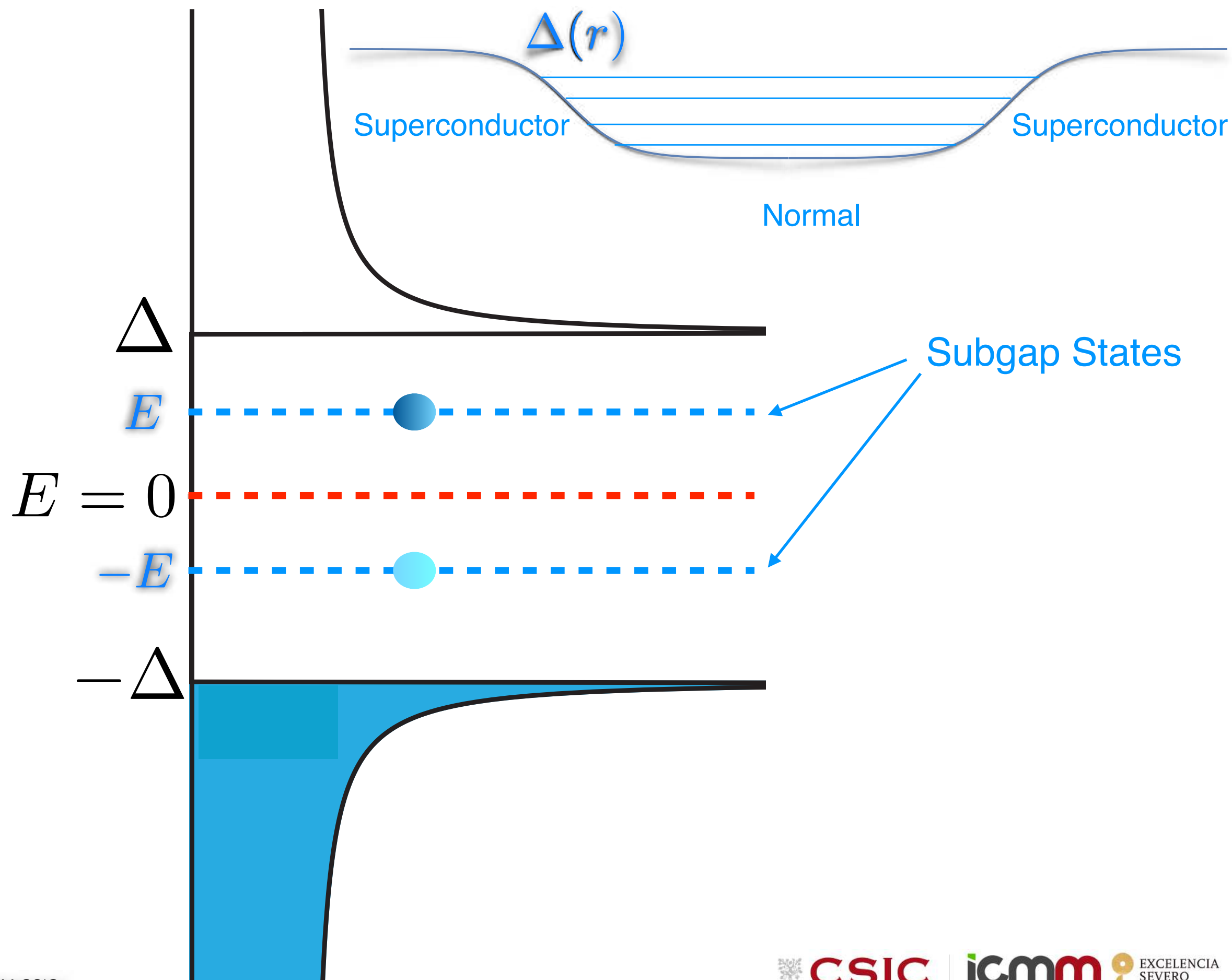
University of Illinois at Urbana-Champaign, 1110 W. Green St., Urbana, IL 61801, USA

Published August 22, 2011

ALL PROPOSALS ARE BASED ON SYSTEMS INVOLVING **SUPERCONDUCTORS, WHY?**





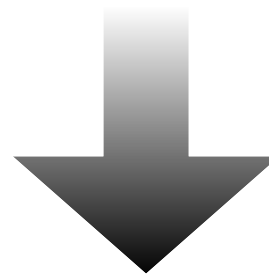


Superconducting BCS mean-field pairing for a generic Hamiltonian

$$\mathcal{H} = \int d\mathbf{r} \left[\sum_{\sigma, \sigma'} H_0^{\sigma, \sigma'}(\mathbf{r}) c_{\sigma \mathbf{r}}^\dagger c_{\sigma' \mathbf{r}} + (\Delta(\mathbf{r}) c_{\uparrow \mathbf{r}}^\dagger c_{\downarrow \mathbf{r}}^\dagger + H.c.) \right].$$

Four-component Nambu spinor

$$\hat{\Psi}(\mathbf{r}) = \begin{pmatrix} c_{\uparrow, \mathbf{r}} \\ c_{\downarrow, \mathbf{r}} \\ -c_{\downarrow, \mathbf{r}}^\dagger \\ c_{\uparrow, \mathbf{r}}^\dagger \end{pmatrix}$$



Bogoliubov-DeGennes (BdG) hamiltonian, which is a **redundant** description to treat electrons and holes at the same footing

$$\mathcal{H} = \frac{1}{2} \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) H_{BdG}(\mathbf{r}) \hat{\Psi}(\mathbf{r}),$$

Bogoliubov-DeGennes (BdG) hamiltonian, which is a **redundant** description to treat electrons and holes at the same footing

$$\mathcal{H} = \frac{1}{2} \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) H_{BdG}(\mathbf{r}) \hat{\Psi}(\mathbf{r}),$$

Electron sector

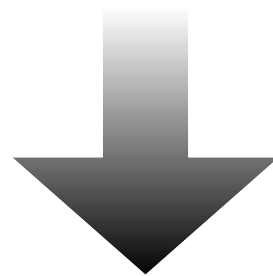
$$H_{BdG}(\mathbf{r}) = \begin{pmatrix} \boxed{H_0(\mathbf{r})} & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & \boxed{-\sigma^y H_0^*(\mathbf{r}) \sigma^y} \end{pmatrix}$$

hole sector (time-reversed
version of the electron)

The redundancy is resolved by observing that the BdG Hamiltonian has **particle-hole (charge conjugation) symmetry**.

$$C = \tau^y \sigma^y = \begin{pmatrix} 0 & -i\sigma^y \\ i\sigma^y & 0 \end{pmatrix}$$

Charge conjugation operator



$$C H_{BdG}^*(r) C^\dagger = -H_{BdG}(r)$$

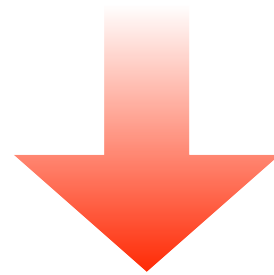
For each eigenstate at positive energy there is an **equivalent particle-hole conjugate eigenstate at negative energy**

Bogoliubov-DeGennes (BdG) hamiltonian, which is a **redundant** description to treat electrons and holes at the same footing

$$H_{BdG}(\mathbf{r})\Phi_n(\mathbf{r}) = E\Phi_n(\mathbf{r}),$$

$$\Phi_n(\mathbf{r}) = [u_{n\uparrow}(\mathbf{r}), u_{n\downarrow}(\mathbf{r}), v_{n\uparrow}(\mathbf{r}), v_{n\downarrow}(\mathbf{r})]^T$$

Stationary solutions



$$\mathcal{H} = \frac{1}{2} \sum_n E_n \gamma_n^\dagger \gamma_n$$

$$\gamma_n = \int d\mathbf{r} \Phi_n^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) = \int d\mathbf{r} [u_{n,\uparrow}^*(\mathbf{r}) c_{\mathbf{r},\uparrow} + u_{n,\downarrow}^*(\mathbf{r}) c_{\mathbf{r},\downarrow} - v_{n,\uparrow}^*(\mathbf{r}) c_{\mathbf{r},\downarrow}^\dagger + v_{n,\downarrow}^*(\mathbf{r}) c_{\mathbf{r},\uparrow}^\dagger]$$

$$\mathcal{H} = \frac{1}{2} \sum_n E_n \gamma_n^\dagger \gamma_n$$

$$\gamma_n = \int d\mathbf{r} \Phi_n^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) = \int d\mathbf{r} [u_{n,\uparrow}^*(\mathbf{r}) c_{\mathbf{r},\uparrow} + u_{n,\downarrow}^*(\mathbf{r}) c_{\mathbf{r},\downarrow} - v_{n,\uparrow}^*(\mathbf{r}) c_{\mathbf{r},\downarrow}^\dagger + v_{n,\downarrow}^*(\mathbf{r}) c_{\mathbf{r},\uparrow}^\dagger]$$

Redundancy in the BdG Hamiltonian

Same excitation

$$\gamma_n^\dagger$$

Positive energy

$$E_n$$

$$\gamma_m^\dagger = \gamma_n$$

Negative energy

$$E_m = -E_n$$

$$\Phi_{E_m}(\mathbf{r}) = C \Phi_{E_n}^*(\mathbf{r})$$

$$\begin{aligned}\Phi_{E_m}(\mathbf{r}) &= C\Phi_{E_n}^*(\mathbf{r}) \\ \gamma_m^\dagger &= \gamma_n \\ E_m &= -E_n = E\end{aligned}$$

- These symmetries originate from the Nambu spinor whose form is in turn dictated by the structure of the second quantized BdG Hamiltonian. **BdG theory has Majorana character built in!**

$$\hat{\Psi}(\mathbf{r}) = C\hat{\Psi}^*(\mathbf{r}) = \begin{pmatrix} \hat{\psi}(\mathbf{r}) \\ i\sigma^y \hat{\psi}^*(\mathbf{r}) \end{pmatrix}$$

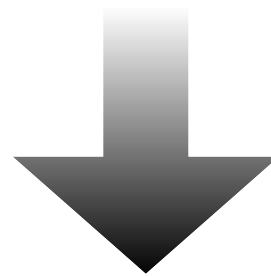
- Note however that this Majorana property is fulfilled by the entire field operator **not** by the eigenmodes of well defined energy (which is what we typically probe in an experiment).

Standard BdG eigenmode

$$\Phi_{E_m}(\mathbf{r}) = C \Phi_{E_n}^*(\mathbf{r})$$

$$\gamma_m^\dagger = \gamma_n$$

$$E_m = -E_n = E$$



$$\gamma_m^\dagger \text{ zero mode} = \gamma_m$$

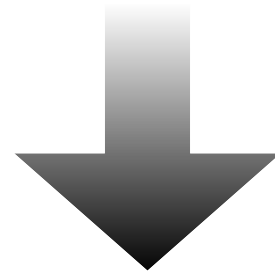
$$E_m = -E_m = 0$$

$$\Phi_0(\mathbf{r}) = C \Phi_0^*(\mathbf{r})$$

$$\Phi_0(\mathbf{r}) = C\Phi_0^*(\mathbf{r})$$

$$\gamma_0^\dagger = \gamma_0$$

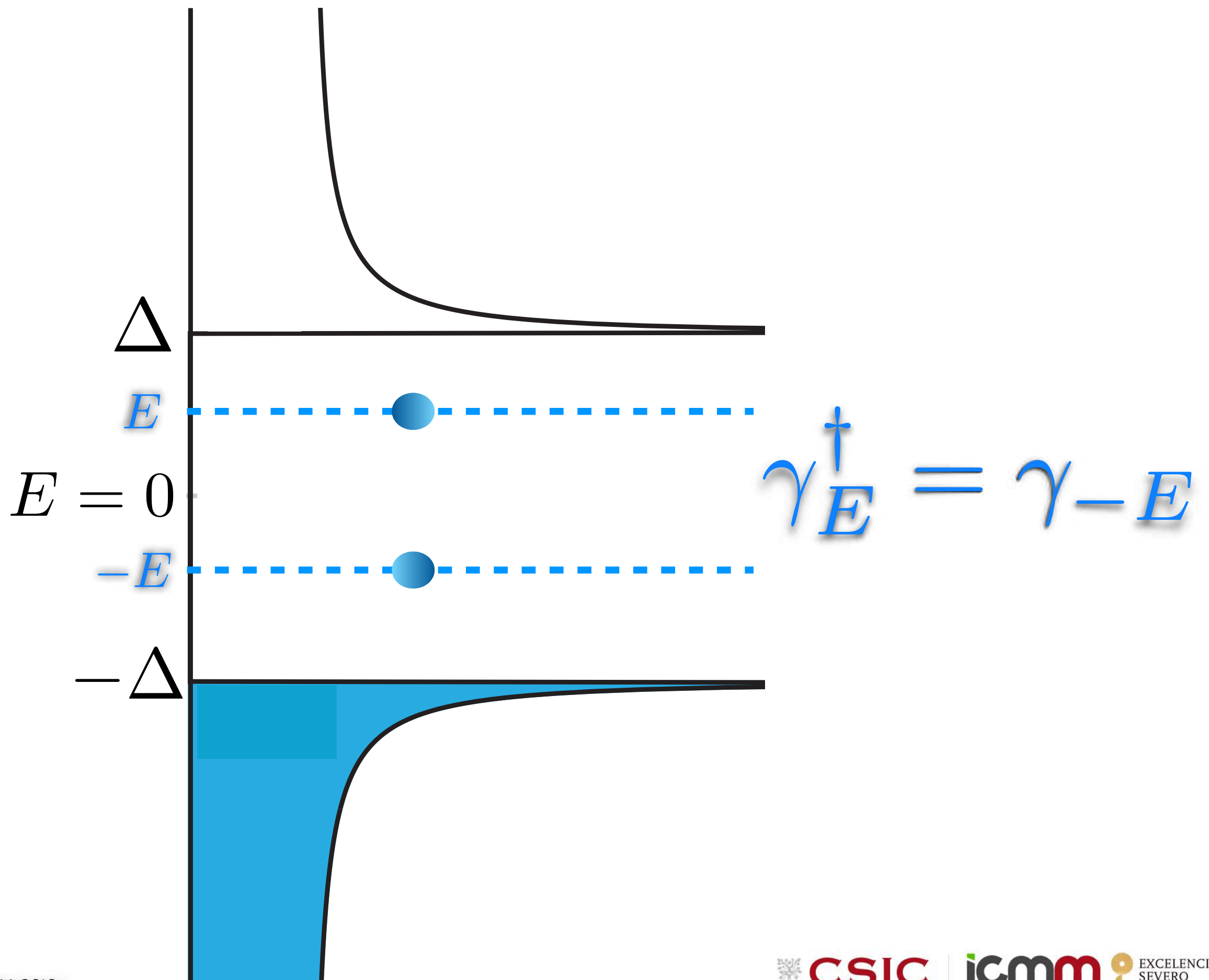
$$E_0 = 0$$

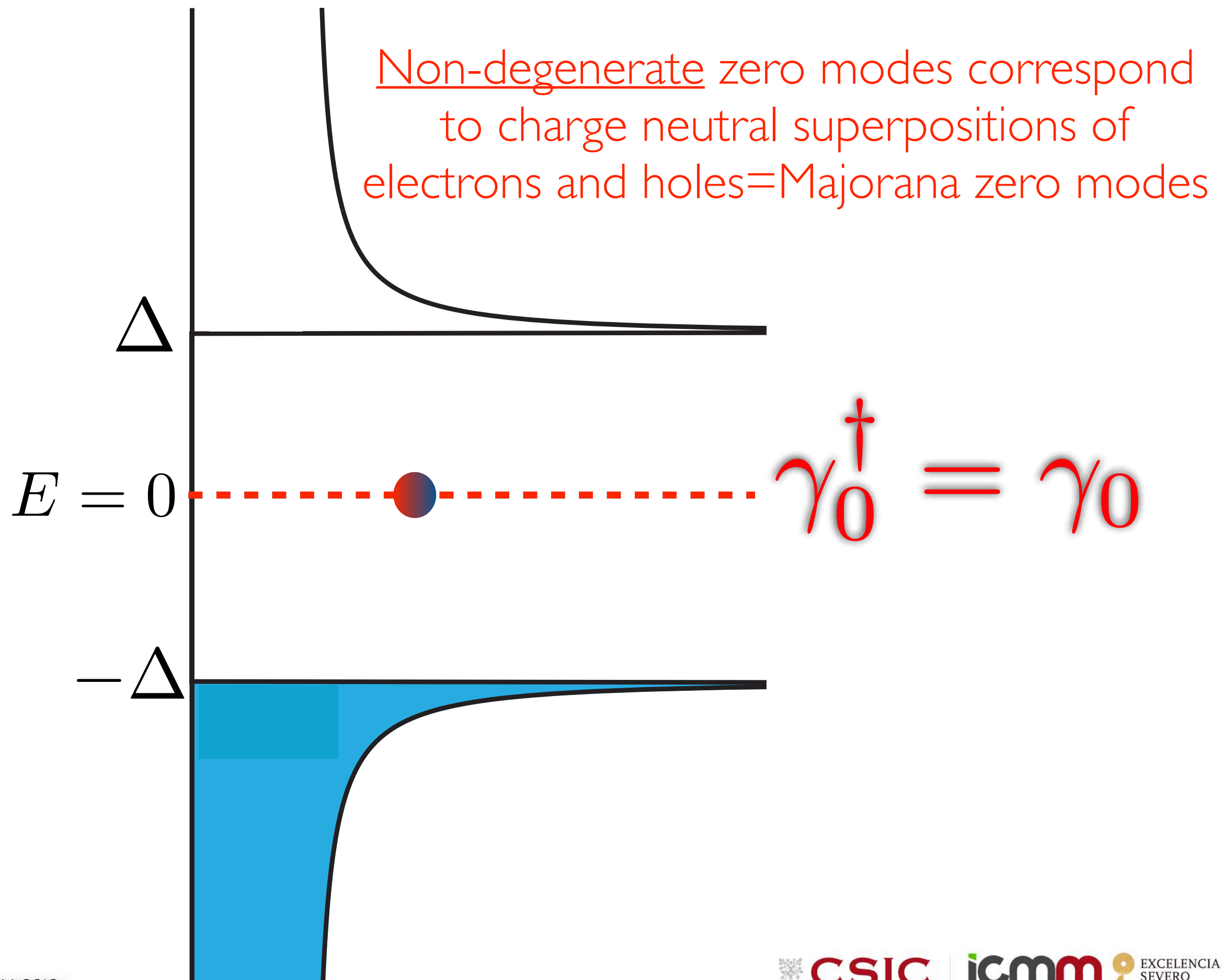


$$\gamma_0 = i \int d\mathbf{r} [u_{0,\uparrow}^*(\mathbf{r})c_{\mathbf{r},\uparrow} + u_{0,\downarrow}^*(\mathbf{r})c_{\mathbf{r},\downarrow} - u_{0,\downarrow}(\mathbf{r})c_{\mathbf{r},\downarrow}^\dagger - u_{0,\uparrow}(\mathbf{r})c_{\mathbf{r},\uparrow}^\dagger]$$

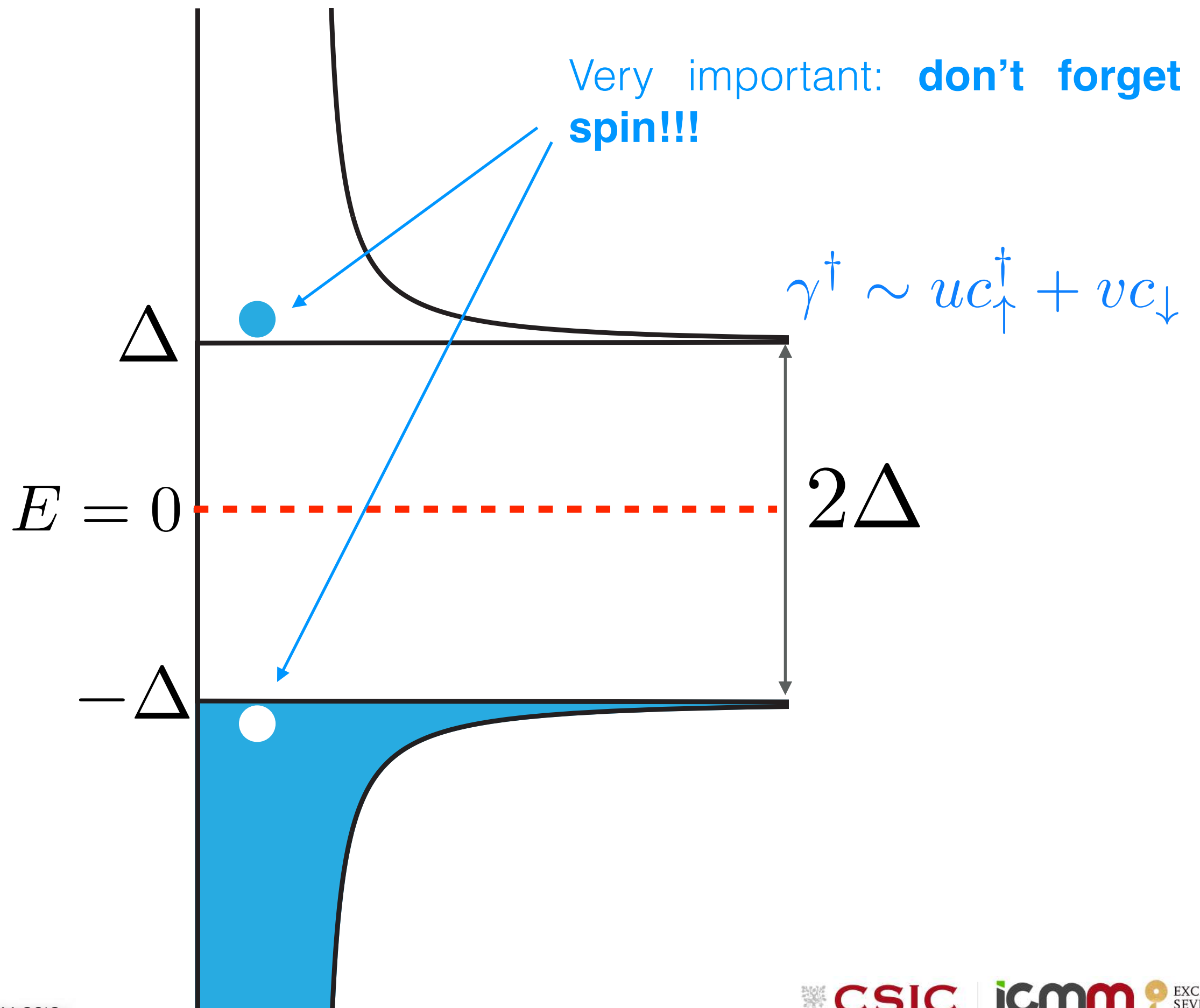
Which is clearly self-conjugate

Majorana Zero Modes

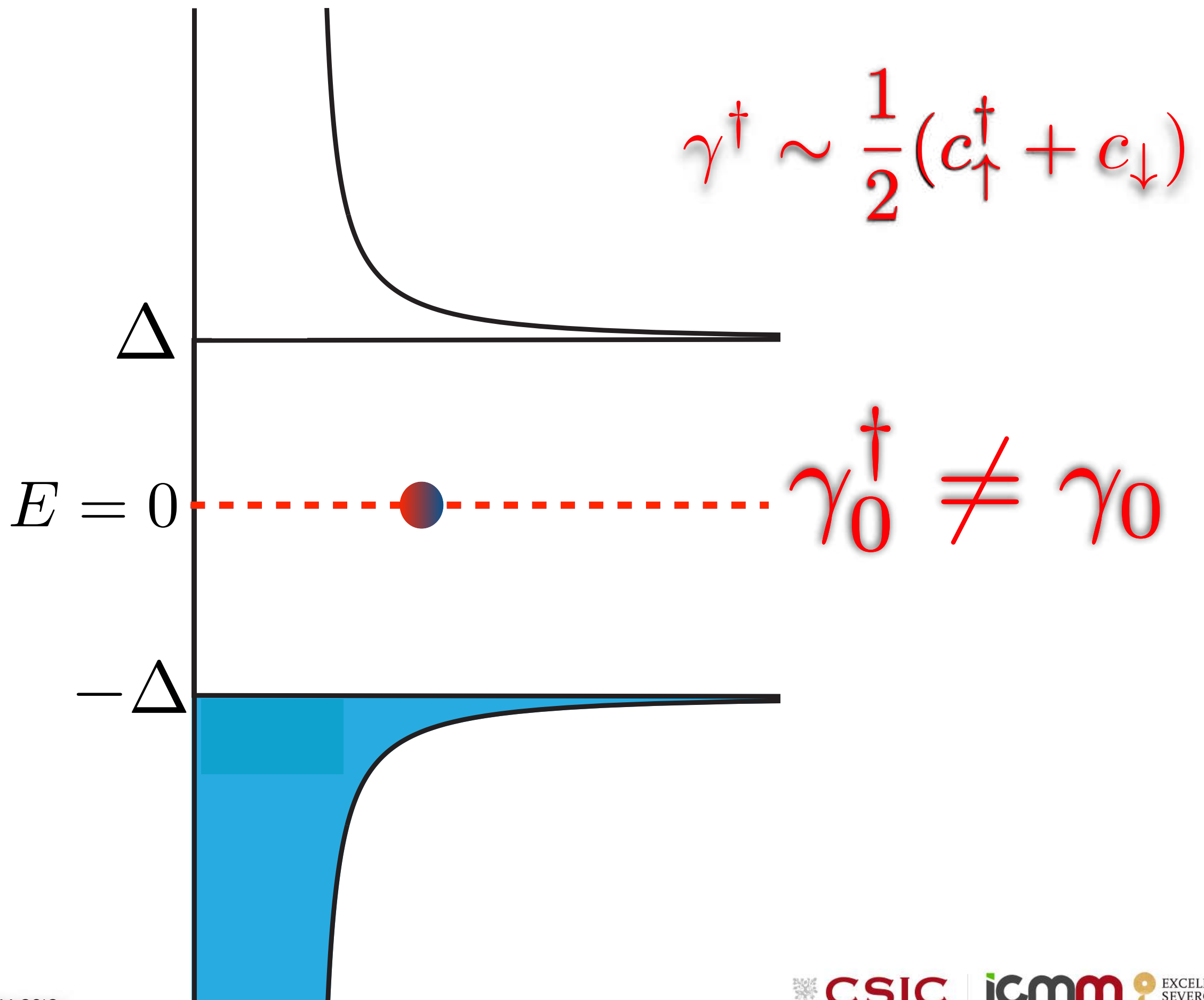


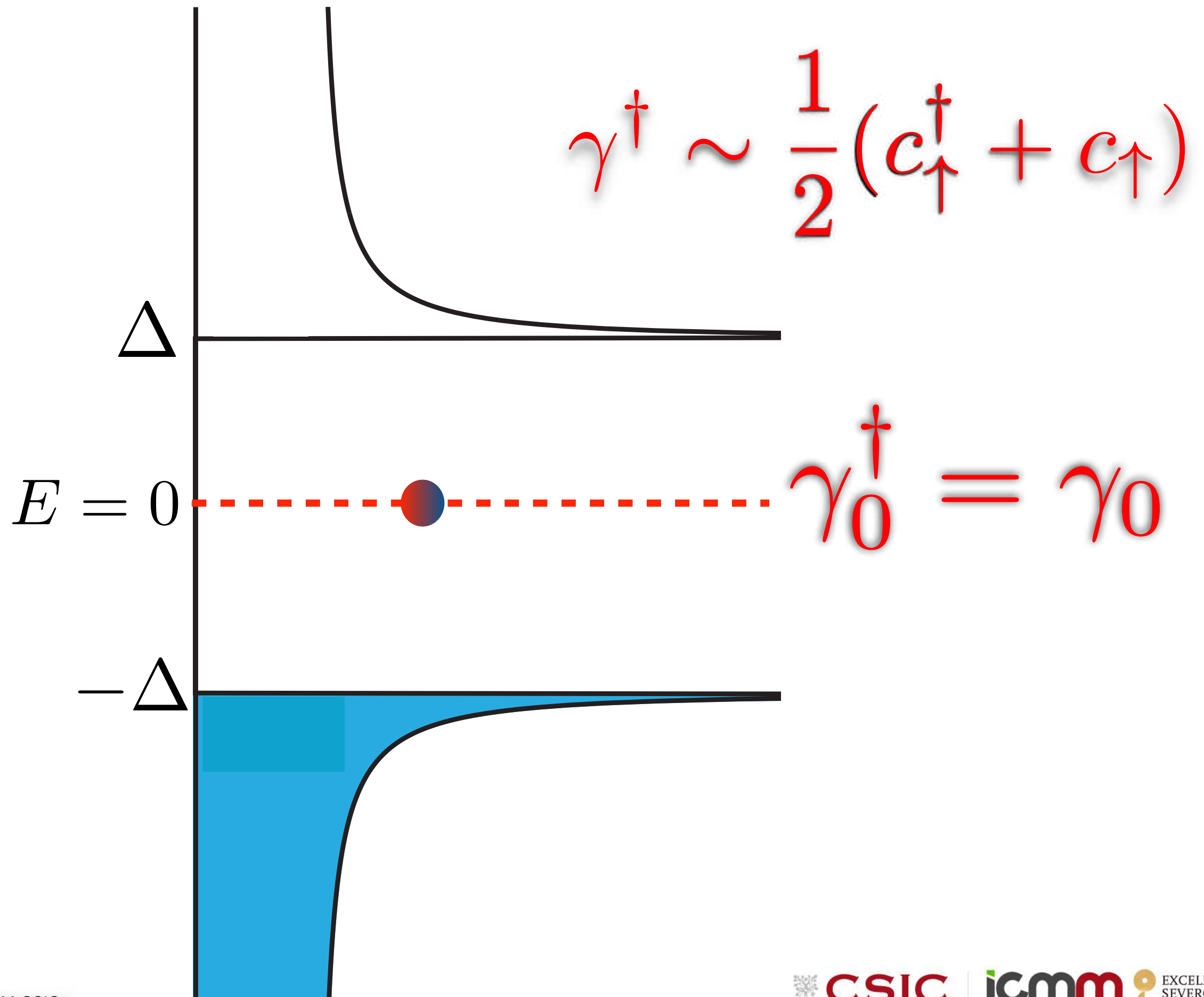


Majorana Zero Modes

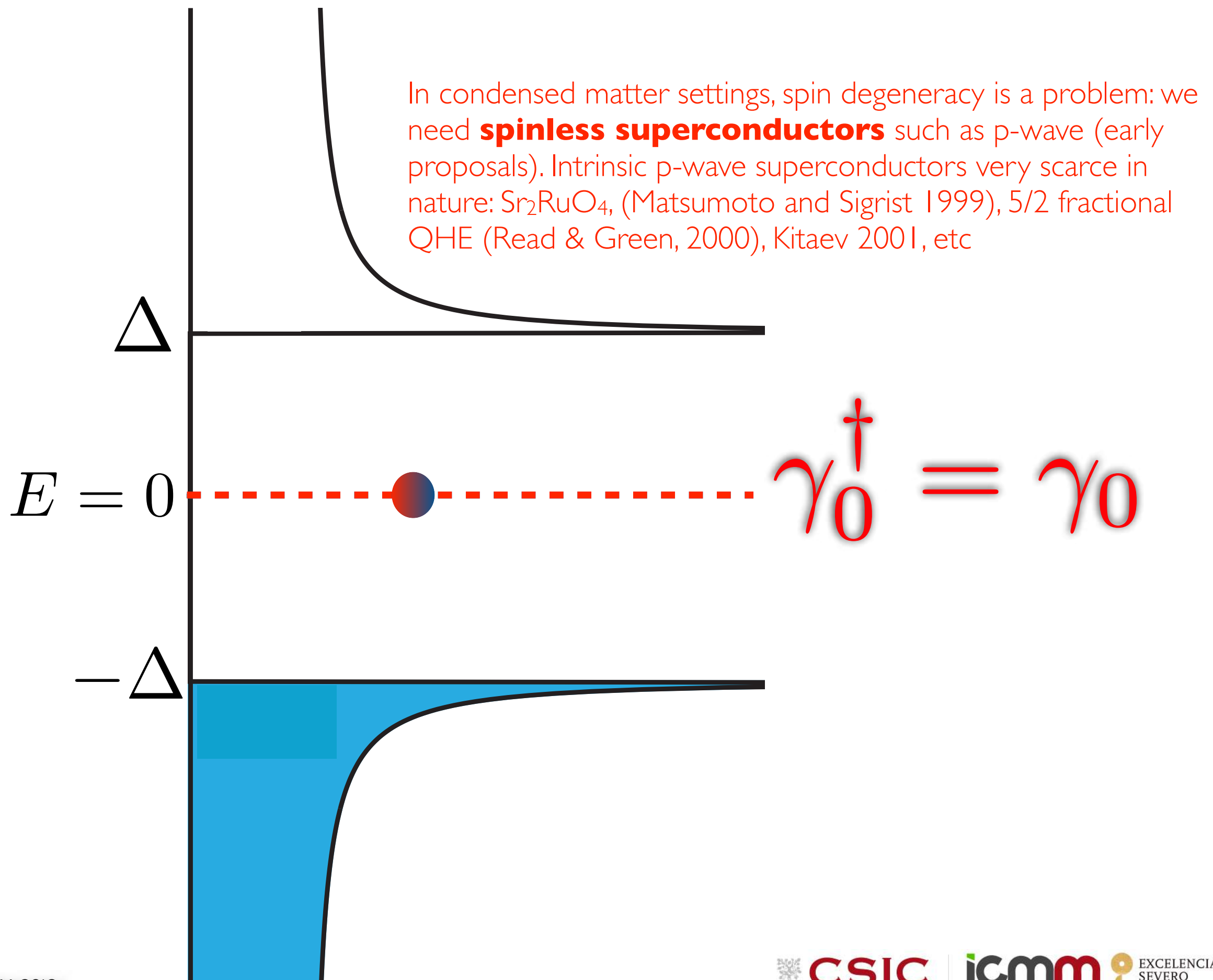


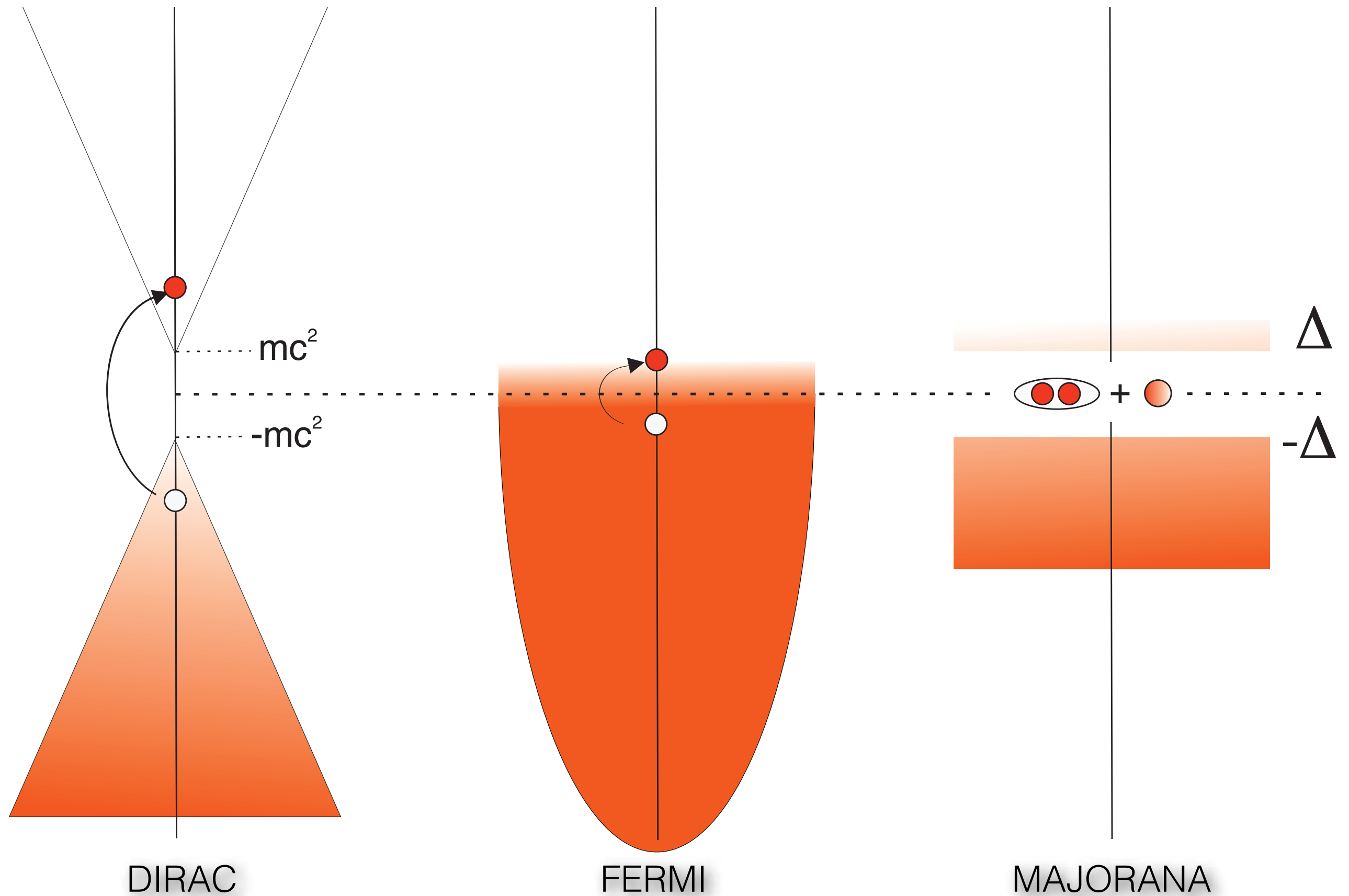
Majorana Zero Modes





Majorana Fermions in Superconductors





Seminal result in field theory (Jackiw and Rossi, 1981): the Majorana equation in two dimensions has non-propagating solutions, namely **zero modes**, if the mass term has a **vortex** profile.

$$\begin{pmatrix} \sigma \cdot p & m \\ m & -\sigma \cdot p \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = i\partial_t \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

$$\sigma \cdot p \psi + i m(\vec{r}) \sigma_y \psi^* = E \psi$$

Vortex in 2D

For an introduction see: “Fractional and Majorana fermions”, Jackiw, arXiv1104.4486

Dirac equation in Weyl representation (chiral spinors where the spin of a particle points in the same direction as its momentum)

$$\begin{pmatrix} -m & i\partial_t - \boldsymbol{\sigma}\mathbf{p} \\ i\partial_t + \boldsymbol{\sigma}\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

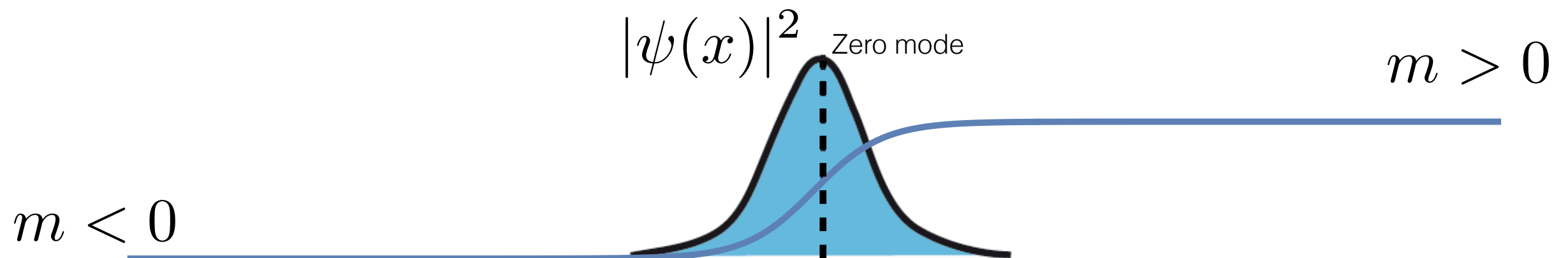
Impose charge conjugation $\Psi^C = \Psi$

$$\begin{aligned} (i\partial_t - \boldsymbol{\sigma}\mathbf{p})\psi_R - im\sigma_y\psi_R^* &= 0 \\ (i\partial_t + \boldsymbol{\sigma}\mathbf{p})\psi_L - im\sigma_y\psi_L^* &= 0 \end{aligned}$$

In the Majorana equation, chiralities become decoupled and **the mass term mixes particles and antiparticles** (technically, this implies that the equation is not gauge invariant, hence Majorana particles cannot be coupled to the electromagnetic field and are thus necessarily charge neutral).

This is a generalisation of another seminal result by Jackiw and Rebbi (Phys. Rev. D **13**, 3398, 1976) which demonstrates the existence of zero energy bound states in the one dimensional Dirac equation at points where **the mass term changes sign**.

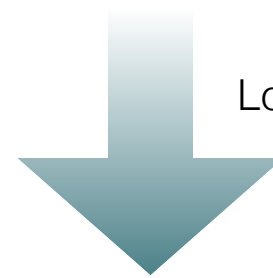
$$[-i\hbar c \partial_x \sigma_x + m(x)c^2 \sigma_z] \psi(x) = E \psi(x)$$



For an introduction see: “Fractional and Majorana fermions”, Jackiw, arXiv1104.4486

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{BdG} = \begin{pmatrix} \frac{p_x^2 + p_y^2}{2m} - \mu & -i\Delta(p_x - ip_y) \\ i\Delta^*(p_x + ip_y) & -\frac{p_x^2 + p_y^2}{2m} + \mu \end{pmatrix}$$



Low momentum

$$H_{BdG} = \begin{pmatrix} -\mu & -i\Delta(p_x - ip_y) \\ i\Delta^*(p_x + ip_y) & \mu \end{pmatrix}$$

N. Read and D. Green, "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect". Phys. Rev. B **61**, 10267 (2000)

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{BdG} = \begin{pmatrix} -\mu & -i\Delta(p_x - ip_y) \\ i\Delta^*(p_x + ip_y) & \mu \end{pmatrix}$$

$$H_{BdG}(p_y = 0) = \Delta p_x \tau^y - \mu \tau^z$$

One-dimensional massive Dirac Hamiltonian (with Pauli matrices defined in electron-hole space). As discussed by Jackiw and Rebbi (Phys. Rev. D **13**, 3398, 1976), this one-dimensional Dirac Hamiltonian contains **zero energy bound states** at interfaces where the mass term changes sign (which is our case as we cross the $\mu = 0$ critical point)

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{BdG} = \begin{pmatrix} -\mu(x) & -i\Delta(-i\partial_x - ip_y) \\ i\Delta^*(i\partial_x + ip_y) & \mu(x) \end{pmatrix}$$

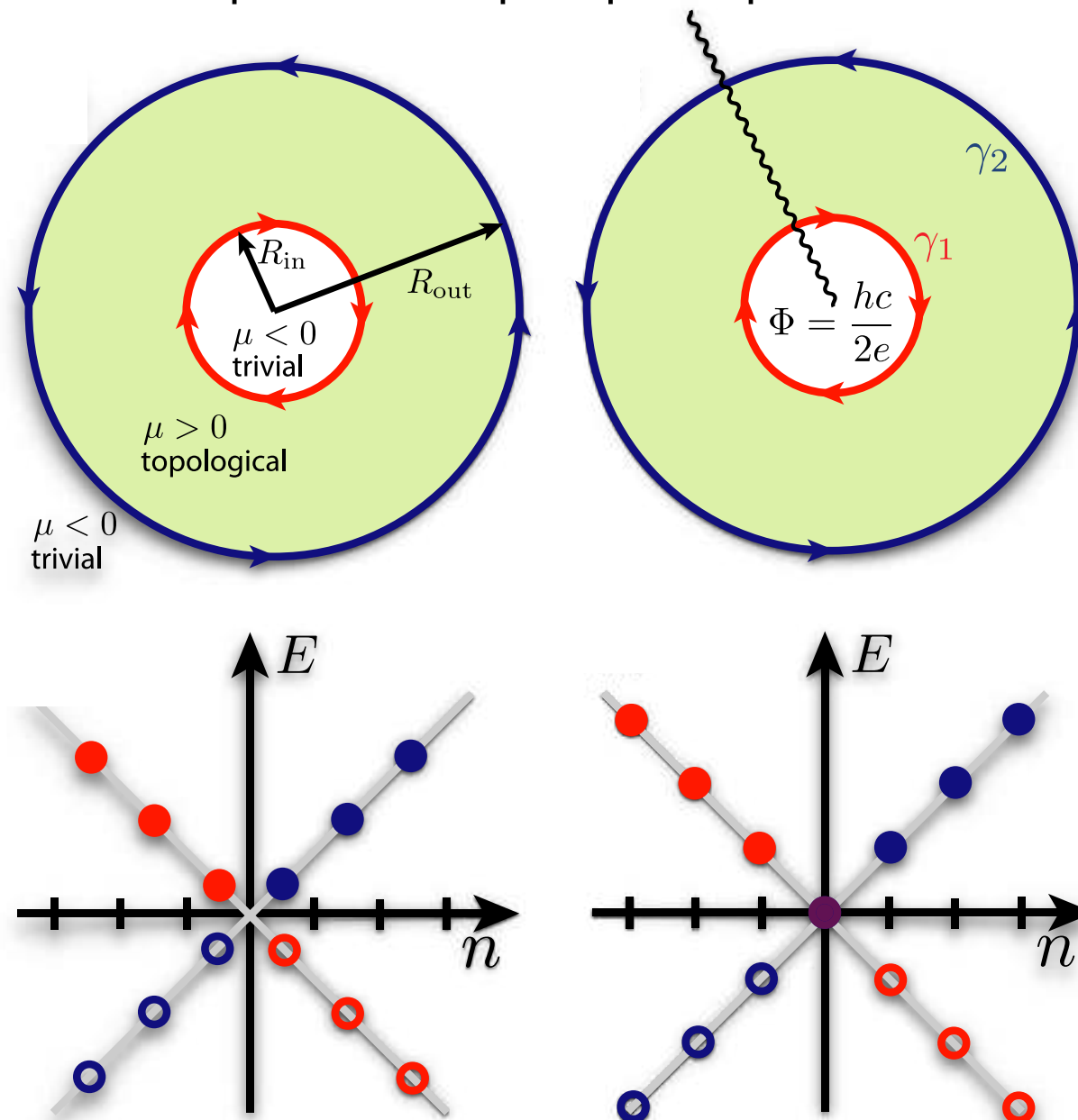
$$\mu(x) = \begin{cases} -\mu_0 < 0, & \text{for } x < 0 \\ \mu_0 > 0, & \text{for } x > 0 \end{cases}$$

$$\Phi(x) \sim \exp \left[- \int_0^x dx' \frac{\mu(x')}{\Delta} \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\gamma \sim \int dx \exp \left[- \int_0^x dx' \frac{\mu(x')}{\Delta} \right] [c(x) + c^\dagger(x)]$$

Zero energy bound states

- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor



Majorana zero modes
bound to vortices

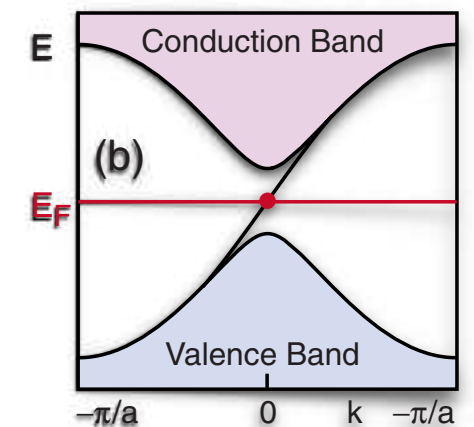
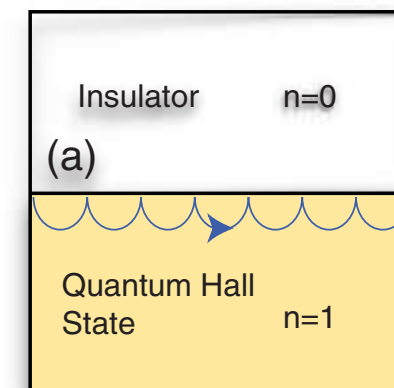
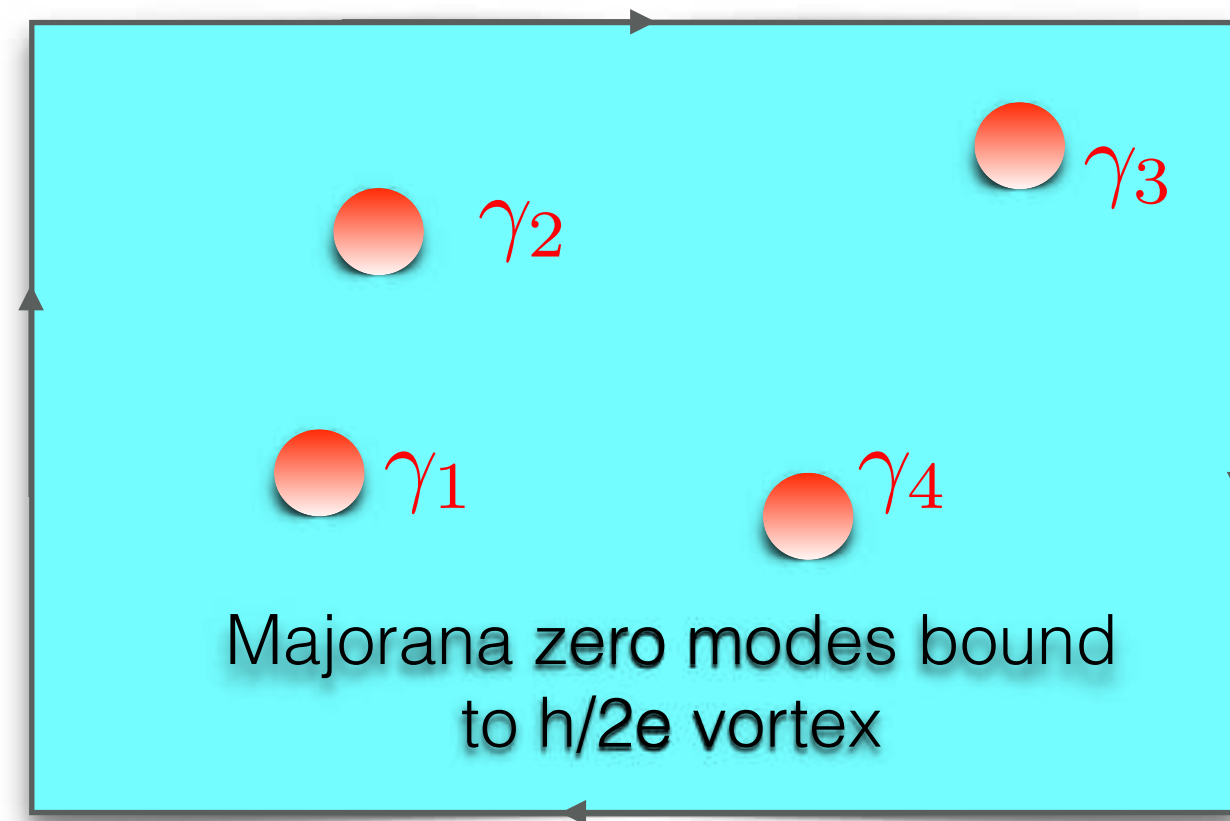


Image from J. Alicea, "New directions in the pursuit of Majorana fermions in solid state systems"
Rep. Prog. Phys. 75, 076501(2012).

First proposals more than 20-30 years old!

Two-dimensional p-wave superconductor

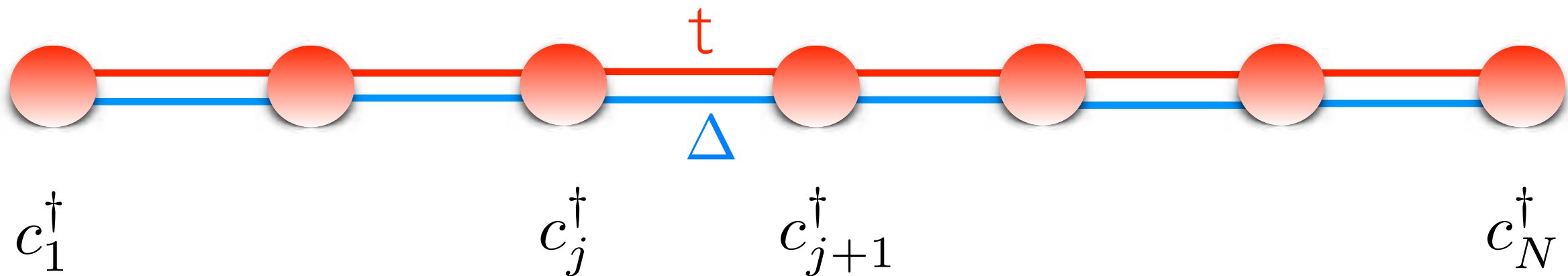


N. Read and D. Green, "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect". Phys. Rev. B **61**, 10267 (2000)

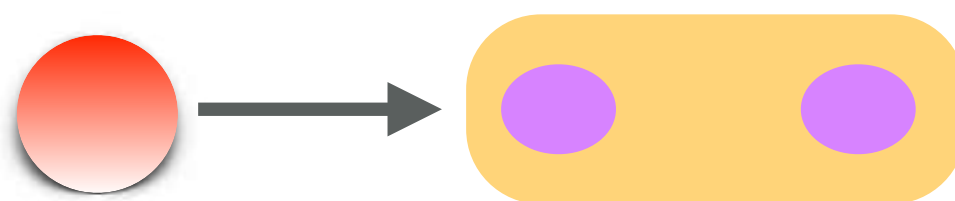
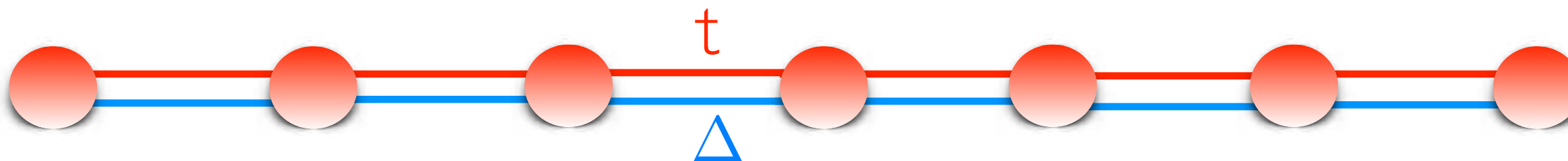
Kitaev, Phys.-Usp., **44**, 131, 2001

$$H = -\mu \sum_{j=1}^N \left(c_j^\dagger c_j - \frac{1}{2} \right) + \sum_{j=1}^{N-1} \left[-t (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \right],$$

1d spinless chain with p-wave pairing



Kitaev, Phys.-Usp., **44**, 131, 2001

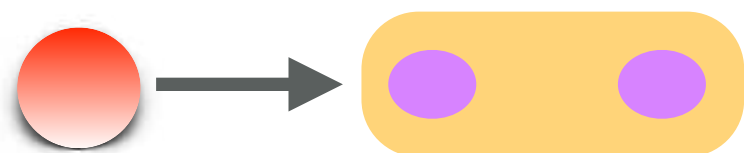
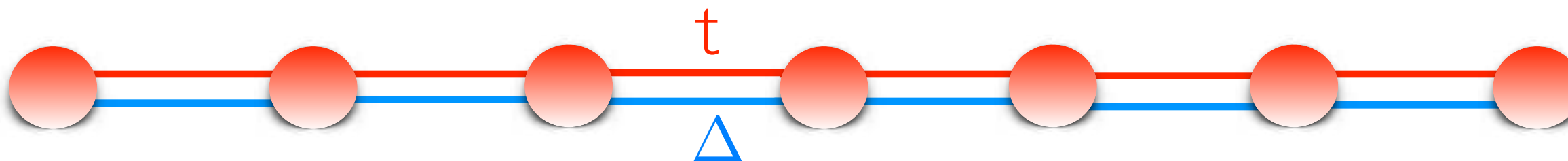


Decomposition in terms of Majorana operators

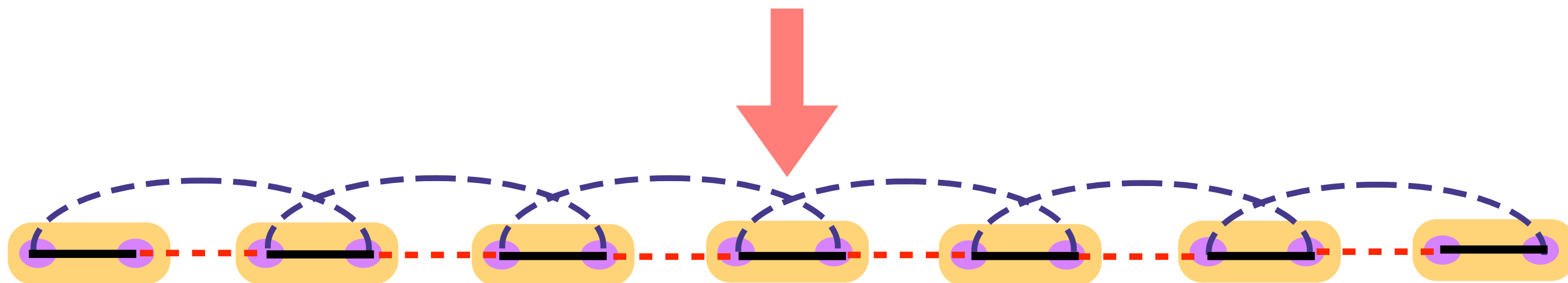
$$c_j = \frac{1}{2} \left(\gamma_j^A + i\gamma_j^B \right), \quad c_j^\dagger = \frac{1}{2} \left(\gamma_j^A - i\gamma_j^B \right).$$

$$\{\gamma_i^A, \gamma_j^B\} = 2\delta_{ij}\delta_{AB}, \quad \gamma_j = \gamma_j^\dagger, \quad \gamma_j^2 = \gamma_j^{\dagger 2} = 1$$

Kitaev, Phys.-Usp., **44**, 131, 2001

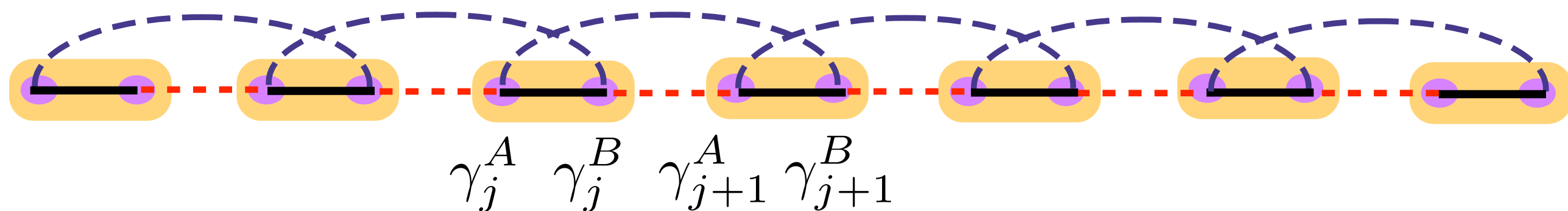


$$\gamma_j^A = c_j + c_j^\dagger, \quad \gamma_j^B = i(c_j^\dagger - c_j)$$

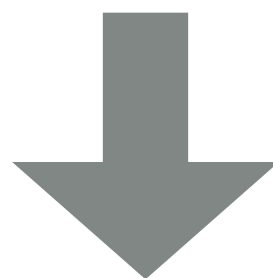


$$\gamma_j^A \quad \gamma_j^B \quad \gamma_{j+1}^A \quad \gamma_{j+1}^B$$

Kitaev, Phys.-Usp., **44**, 131, 2001



$$H = -\mu \sum_{j=1}^N \left(c_j^\dagger c_j - \frac{1}{2} \right) + \sum_{j=1}^{N-1} \left[-t (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \right],$$



$$H = -\frac{i\mu}{2} \sum_{j=1}^N \gamma_j^A \gamma_j^B + \frac{i}{2} \sum_{j=1}^{N-1} \left[\omega_+ \gamma_j^B \gamma_{j+1}^A + \omega_- \gamma_j^A \gamma_{j+1}^B \right],$$

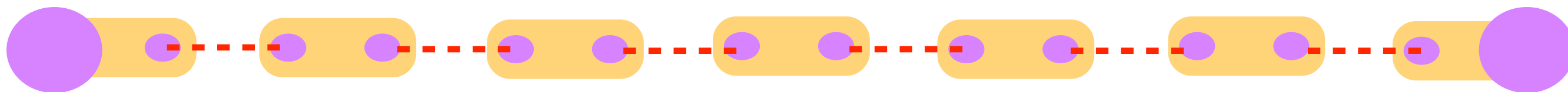
$$\omega_+ = \Delta + t$$

$$\omega_- = \Delta - t$$

$$\mu$$

Kitaev, Phys.-Usp., **44**, 131, 2001

$$\mu = 0 \quad t = \Delta$$



$$H = -\frac{i\mu}{2} \sum_{j=1}^N \gamma_j^A \gamma_j^B + \frac{i}{2} \sum_{j=1}^{N-1} \left[\omega_+ \gamma_j^B \gamma_{j+1}^A + \omega_- \gamma_j^A \gamma_{j+1}^B \right],$$

Despite its innocent-looking form, this model is rather nontrivial:

$$H = it \sum_{j=1}^{N-1} \gamma_j^B \gamma_{j+1}^A.$$

- Majorana operators on the same site are now decoupled
- long range coupling is established since Majorana operators on neighbouring sites are now coupled.
- The Majorana operators at the end of chain seem to have disappeared from the problem...

...

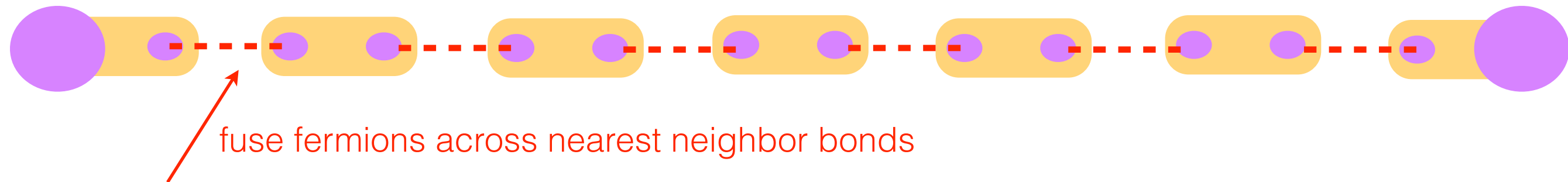
$$\omega_+ = 2t$$

$$\omega_- = 0$$

$$\mu = 0$$

Kitaev, Phys.-Usp., **44**, 131, 2001

$$\mu = 0 \quad t = \Delta$$



$$d_j = \frac{1}{2} \left(\gamma_j^B + i \gamma_{j+1}^A \right)$$

$$d_j^\dagger = \frac{1}{2} \left(\gamma_j^B - i \gamma_{j+1}^A \right).$$

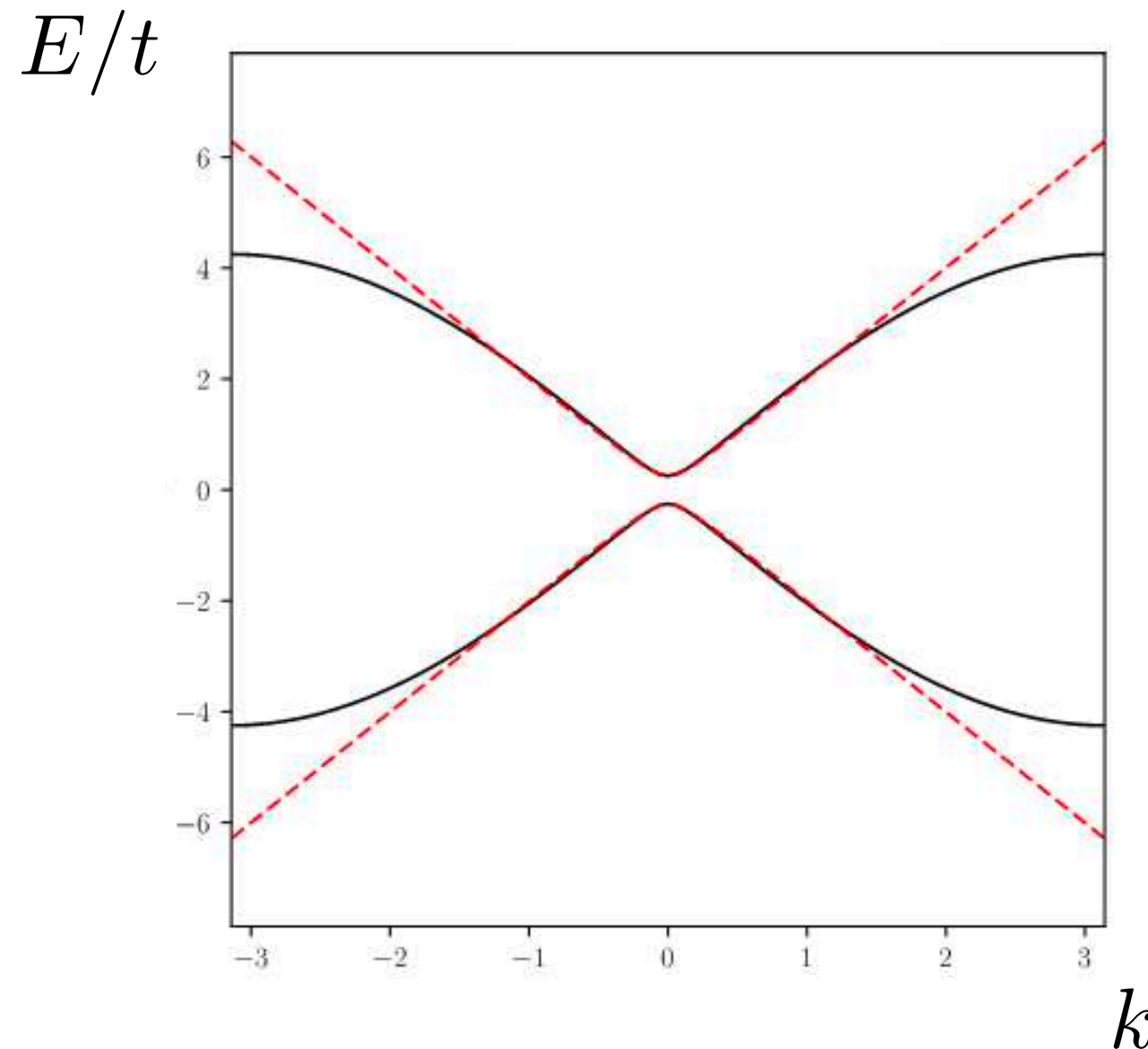
$$H = 2t \sum_{j=1}^{N-1} \left(d_j^\dagger d_j - \frac{1}{2} \right).$$

$$[H, \gamma_1^A] = [H, \gamma_N^B] = 0.$$

GAPPED SPECTRUM+ZERO-ENERGY MAJORANAS AT
THE END OF THE WIRE (DECOUPLED FROM THE
BULK OF THE CHAIN)!!!

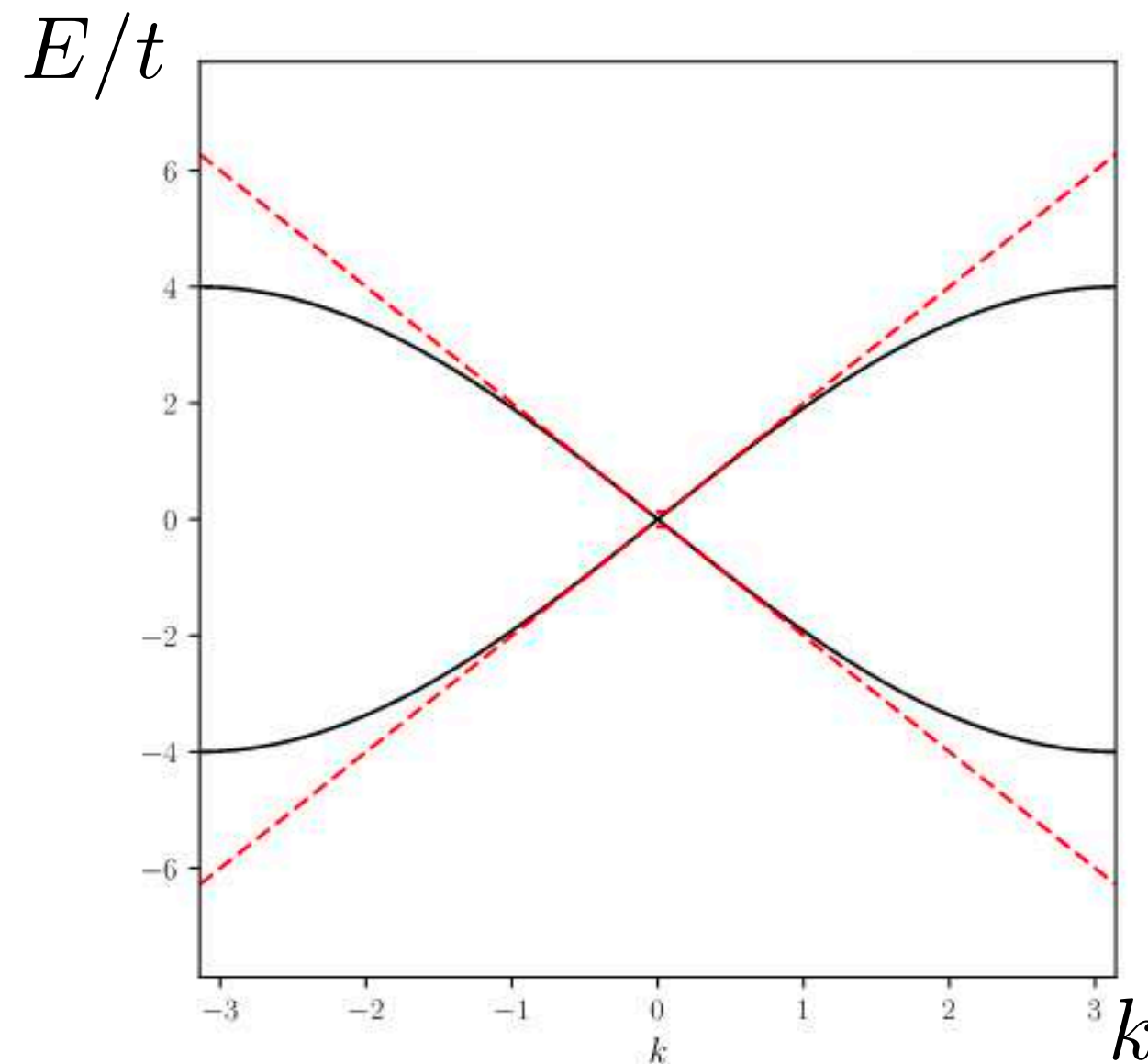
Kitaev, Phys.-Usp., **44**, 131, 2001

$$\mu = -2.25t \quad \text{Trivial Phase}$$

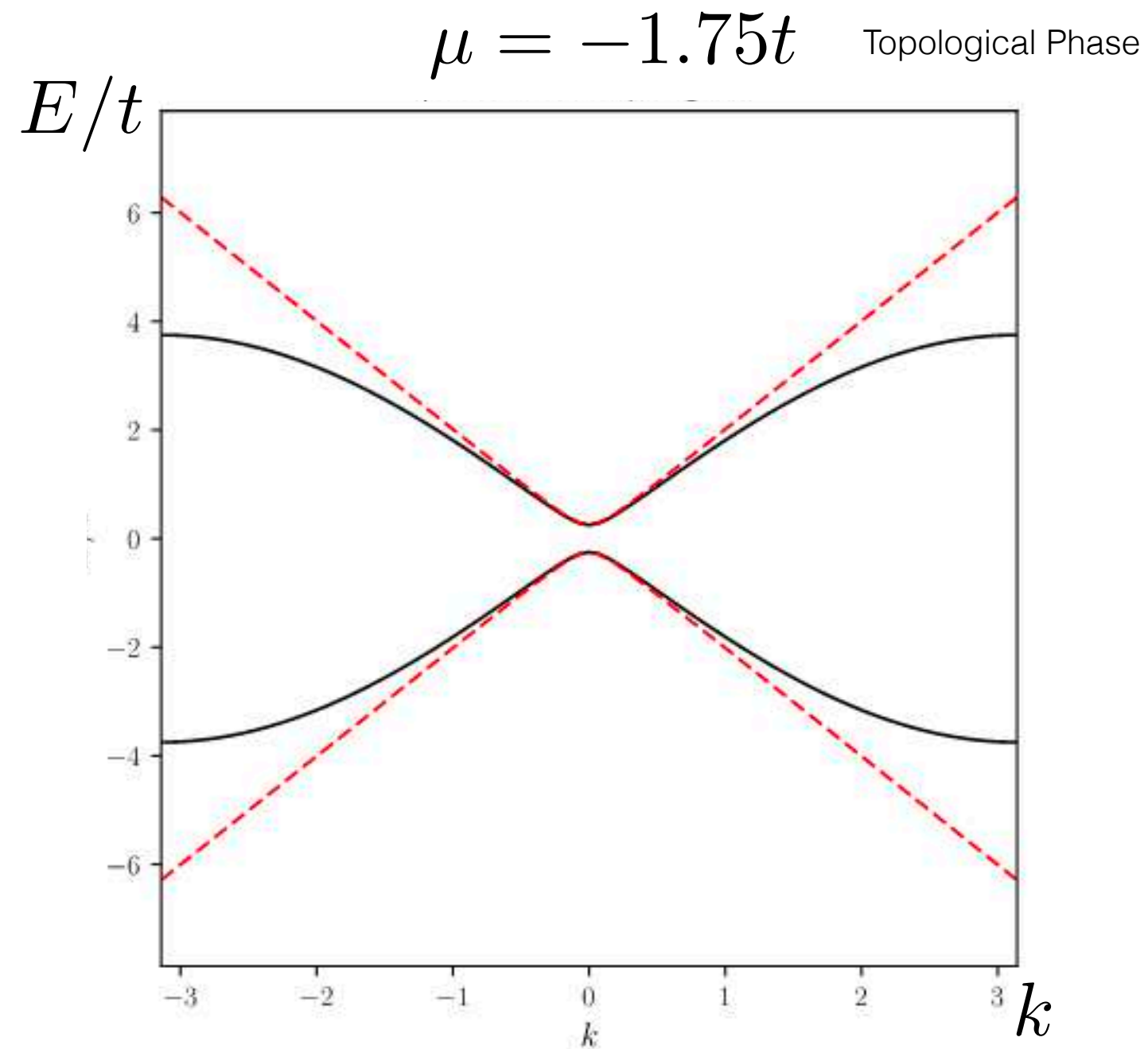


Kitaev, Phys.-Usp., **44**, 131, 2001

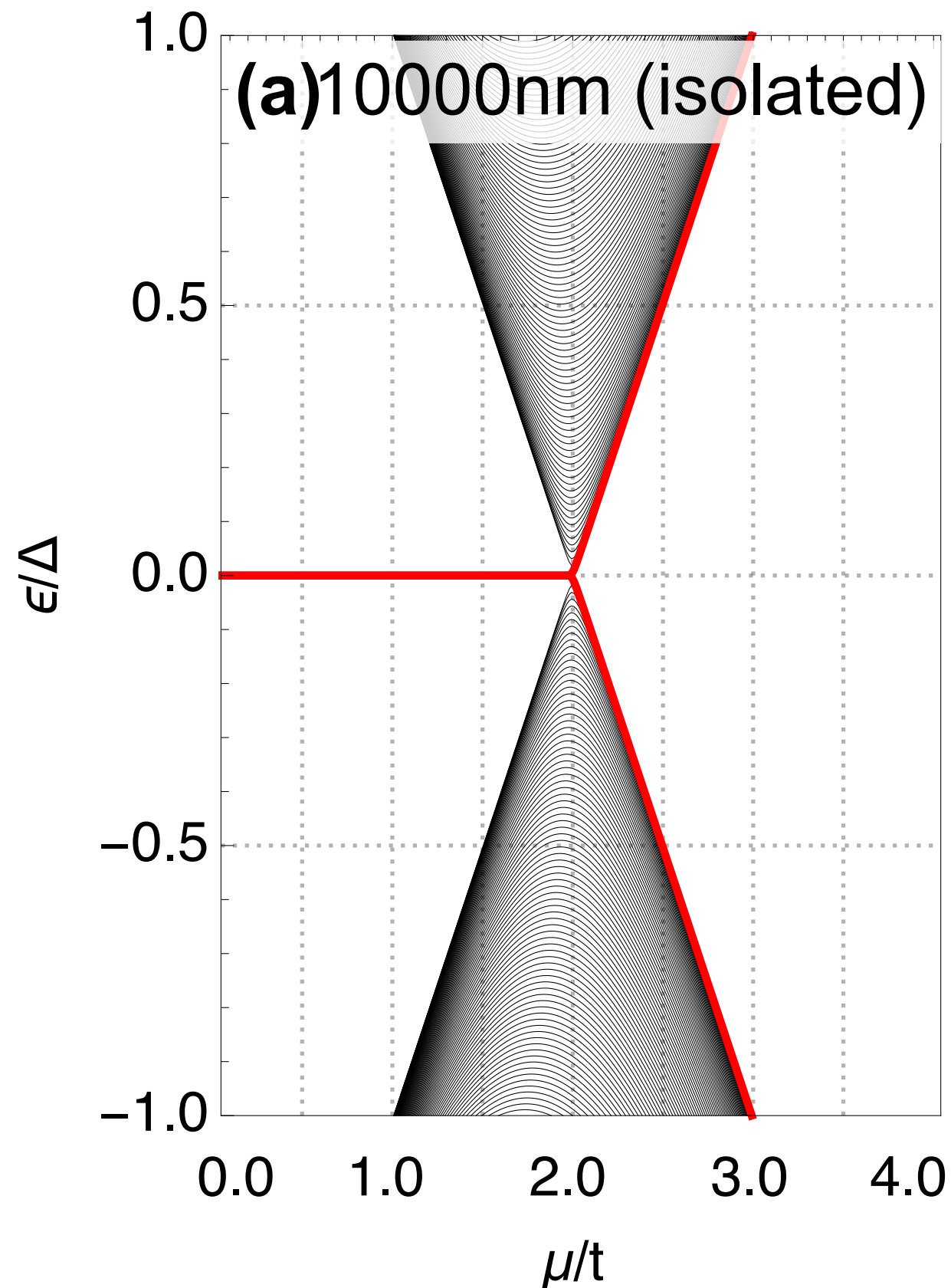
$\mu = -2t$ **Gap closes**



Kitaev, Phys.-Usp., **44**, 131, 2001



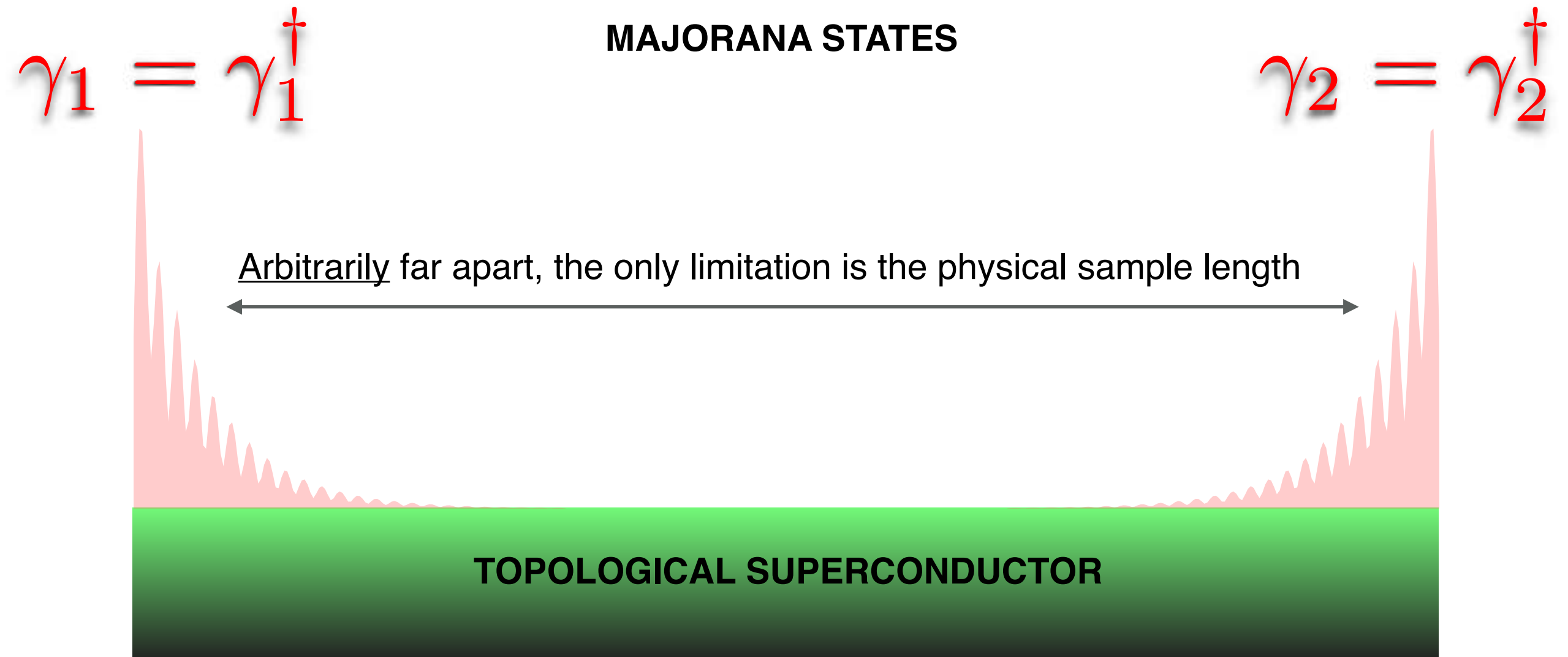
Kitaev, Phys.-Usp., **44**, 131, 2001



The system is topologically non-trivial (topological superconductor) for

$$|\mu| < 2t$$

MAJORANA STATES



Two Majorana quasiparticles at the ends of a one dimensional topological superconductor form a single electronic excitation which can be arbitrarily delocalized.

$$-i\gamma_1\gamma_2 = (1 - 2c^\dagger c)$$



$$|\Psi\rangle \rightarrow |\Psi\rangle$$

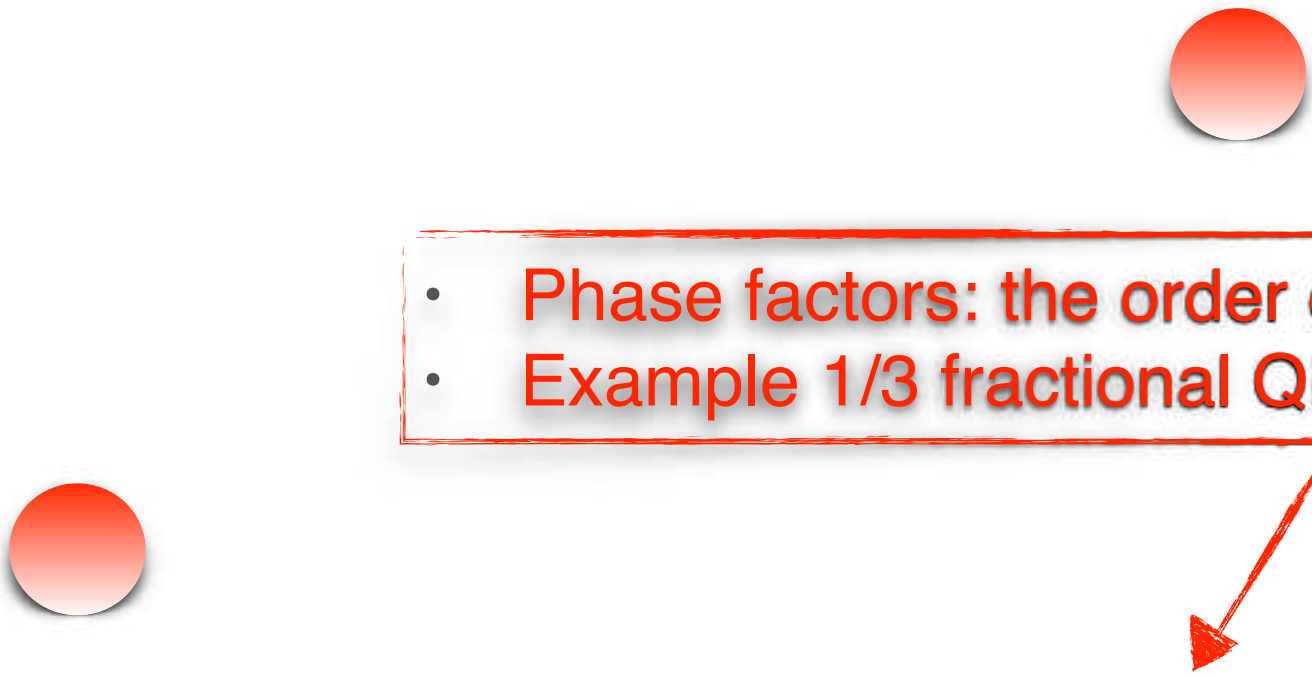
Bosons



$$|\Psi\rangle \rightarrow -|\Psi\rangle$$

Fermions

- Phase factors: the order of the exchange is irrelevant
- Example 1/3 fractional QHE

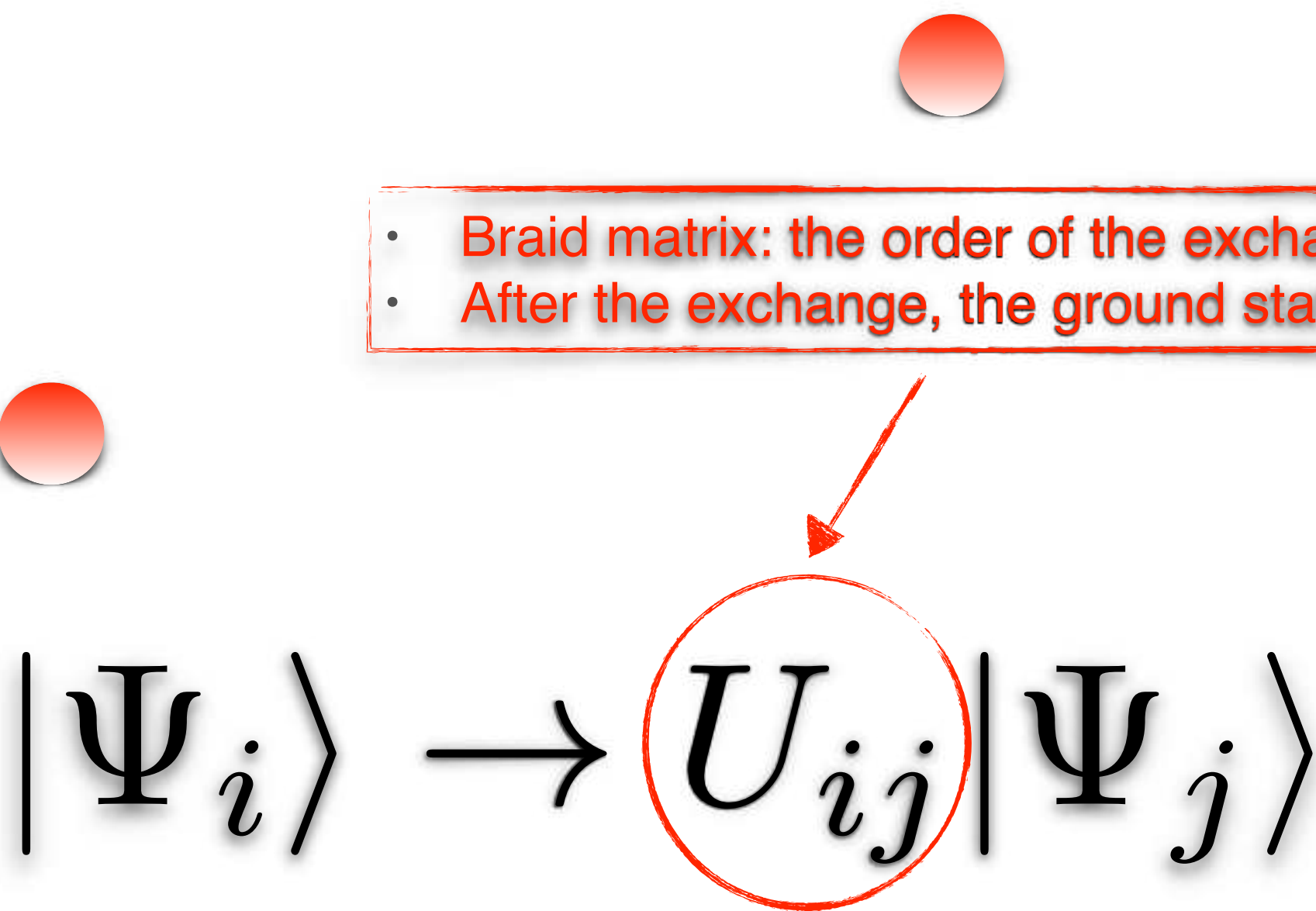

$$|\Psi\rangle \rightarrow e^{i\theta} |\Psi\rangle$$

Abelian Anyons

J. Leinaas, J. Myrheim, Jan, "On the theory of identical particles" *Il Nuovo Cimento B.* **37**, 1, 1977

Frank Wilczek, "Quantum Mechanics of Fractional-Spin Particles", *Phys. Rev. Lett.* **49**, 957, 1982

- Braid matrix: the order of the exchange matters
- After the exchange, the ground state changes


$$|\Psi_i\rangle \rightarrow U_{ij} |\Psi_j\rangle$$

non-Abelian Anyons



γ_1



γ_2

Fermionic parity operator

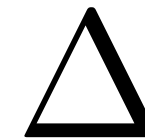
$$P = 1 - 2\hat{n} = 1 - 2c^\dagger c = -i\gamma_1\gamma_2$$

Majorana zero modes encode ground state degeneracy

Majorana zero mode is “half” a Fermion

$$c = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

$$c^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

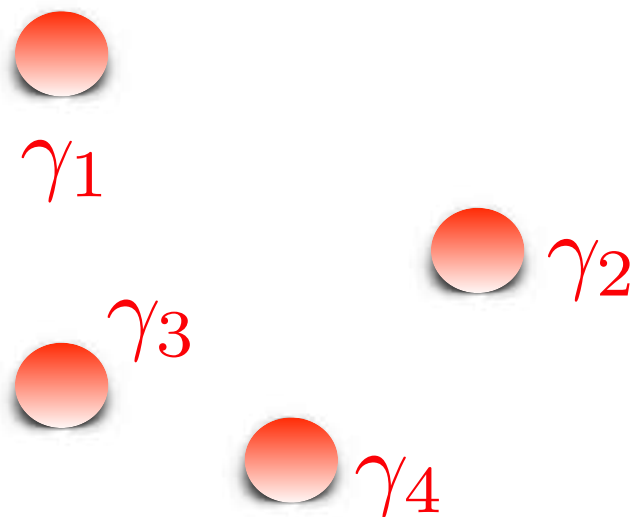


Gap to higher energy excited states

$$|0\rangle \text{ ————— } |1\rangle = c^\dagger |0\rangle$$

Locally indistinguishable ground states

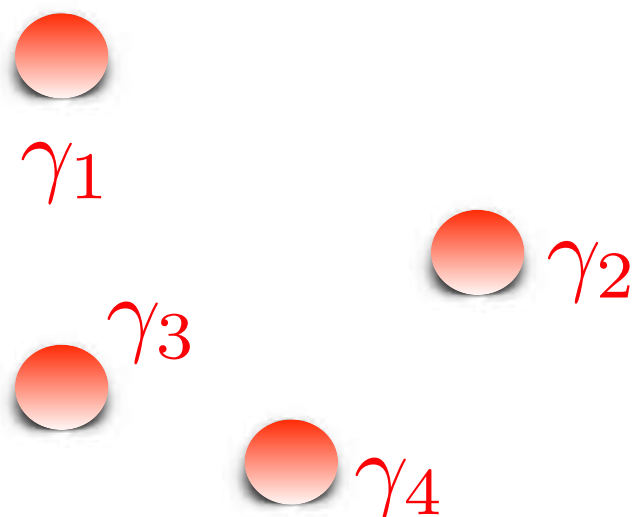
Two fermionic parity ground states



$$c_A = \frac{1}{2}(\gamma_1 + i\gamma_2) \quad c_B = \frac{1}{2}(\gamma_3 + i\gamma_4)$$

$$c_A^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2) \quad c_B^\dagger = \frac{1}{2}(\gamma_3 - i\gamma_4)$$

Two fermionic parity ground states	
$ 0, 0\rangle$ Even $ 1, 1\rangle = c_A^\dagger c_B^\dagger 0, 0\rangle$ $ \uparrow\rangle \equiv 0, 0\rangle$ $ \downarrow\rangle \equiv 1, 1\rangle$	$ 0, 1\rangle = c_B^\dagger 0, 0\rangle$ Odd $ 1, 0\rangle = c_A^\dagger 0, 0\rangle$ $ \uparrow\rangle \equiv 1, 0\rangle$ $ \downarrow\rangle \equiv 0, 1\rangle$



We can define a Majorana qubit by fixing the parity (even or odd) of the total number of fermions and construct the three Pauli matrices using four Majorana zero modes.

$$\sigma_x = -i\gamma_2\gamma_3$$

$$\sigma_y = -i\gamma_1\gamma_3$$

$$\sigma_z = -i\gamma_1\gamma_2,$$

Two fermionic parity ground states	
$ 0, 0\rangle$ Even	$ 0, 1\rangle = c_B^\dagger 0, 0\rangle$ Odd
$ 1, 1\rangle = c_A^\dagger c_B^\dagger 0, 0\rangle$	$ 1, 0\rangle = c_A^\dagger 0, 0\rangle$
$ \uparrow\rangle \equiv 0, 0\rangle$	$ \uparrow\rangle \equiv 1, 0\rangle$
$ \downarrow\rangle \equiv 1, 1\rangle$	$ \downarrow\rangle \equiv 0, 1\rangle$

The qubit stores quantum information non-locally (through the spatially separated Majorana zero modes) which gives rise to the concept of topologically protected (fault-tolerant) quantum computation (immune to local errors/decoherence)

$$U_{12} = e^{-i \frac{\pi}{4} \gamma_1 \gamma_2}$$



γ_1



γ_2

$$\gamma_1 \rightarrow -\gamma_2$$

$$\gamma_2 \rightarrow \gamma_1$$

$$U_{12}|n\rangle = e^{\frac{\pi}{4}(1-2n)}|n\rangle$$

Exchanging two Majorana zero modes generates occupancy dependent phase factors

$$U_{12}|0\rangle = \frac{1}{\sqrt{2}}(1+i)|0\rangle$$

$$U_{12}|1\rangle = \frac{1}{\sqrt{2}}(1-i)|1\rangle$$

Majorana zero modes encode ground state degeneracy



γ_2



γ_2



γ_3



γ_3

Two fermionic parity ground states			
$ 0, 0\rangle$	Even	$ 0, 1\rangle = c_B^\dagger 0, 0\rangle$	Odd
$ 1, 1\rangle = c_A^\dagger c_B^\dagger 0, 0\rangle$		$ 1, 0\rangle = c_A^\dagger 0, 0\rangle$	
$ \uparrow\rangle \equiv 0, 0\rangle$		$ \uparrow\rangle \equiv 1, 0\rangle$	
$ \downarrow\rangle \equiv 1, 1\rangle$		$ \downarrow\rangle \equiv 0, 1\rangle$	

$$U_{12}|n_1, n_2\rangle = e^{\frac{\pi}{4}(1-2n_1)}|n_1, n_2\rangle$$

$$U_{34}|n_1, n_2\rangle = e^{\frac{\pi}{4}(1-2n_2)}|n_1, n_2\rangle,$$

Occupation-dependent phase factors



γ_1



γ_3



γ_3



γ_4

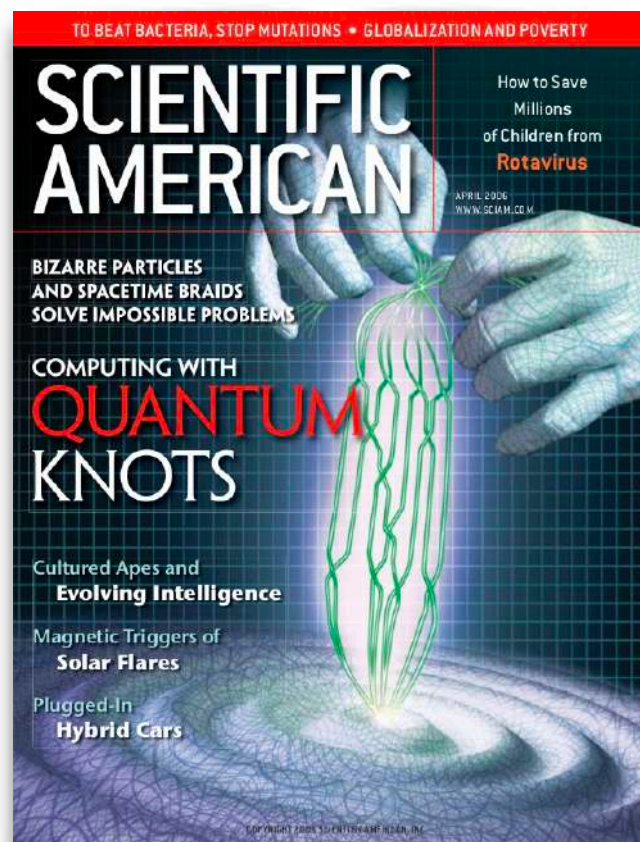
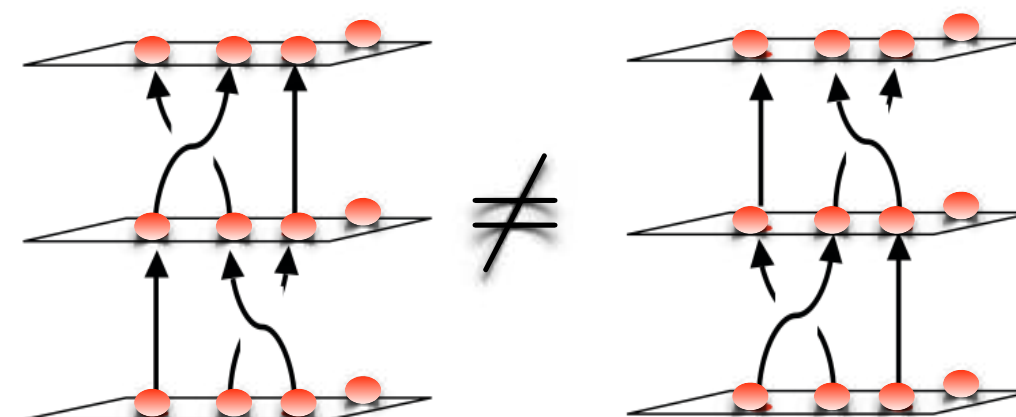
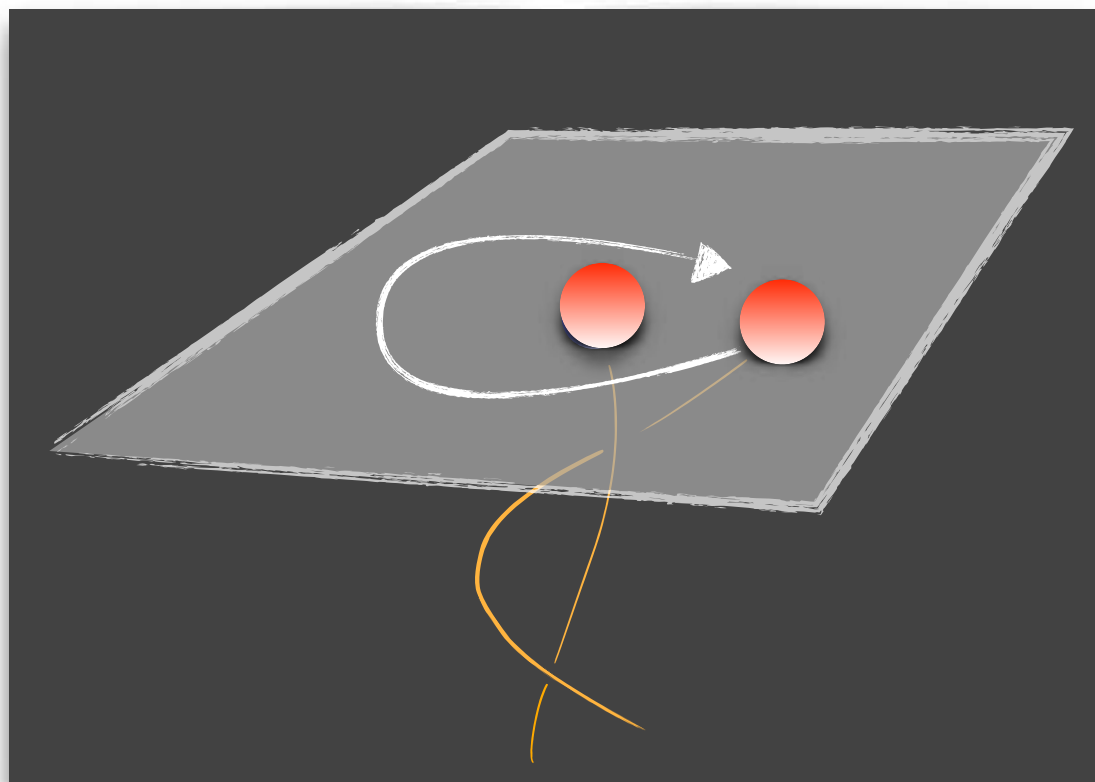
$$U_{23}|n_1, n_2\rangle = \frac{1}{\sqrt{2}} [|n_1, n_2\rangle + i(-1)^{n_1} |1 - n_1, 1 - n_2\rangle].$$

$$U_{23}|0, 0\rangle = \frac{1}{\sqrt{2}} [|0, 0\rangle + i|1, 1\rangle].$$

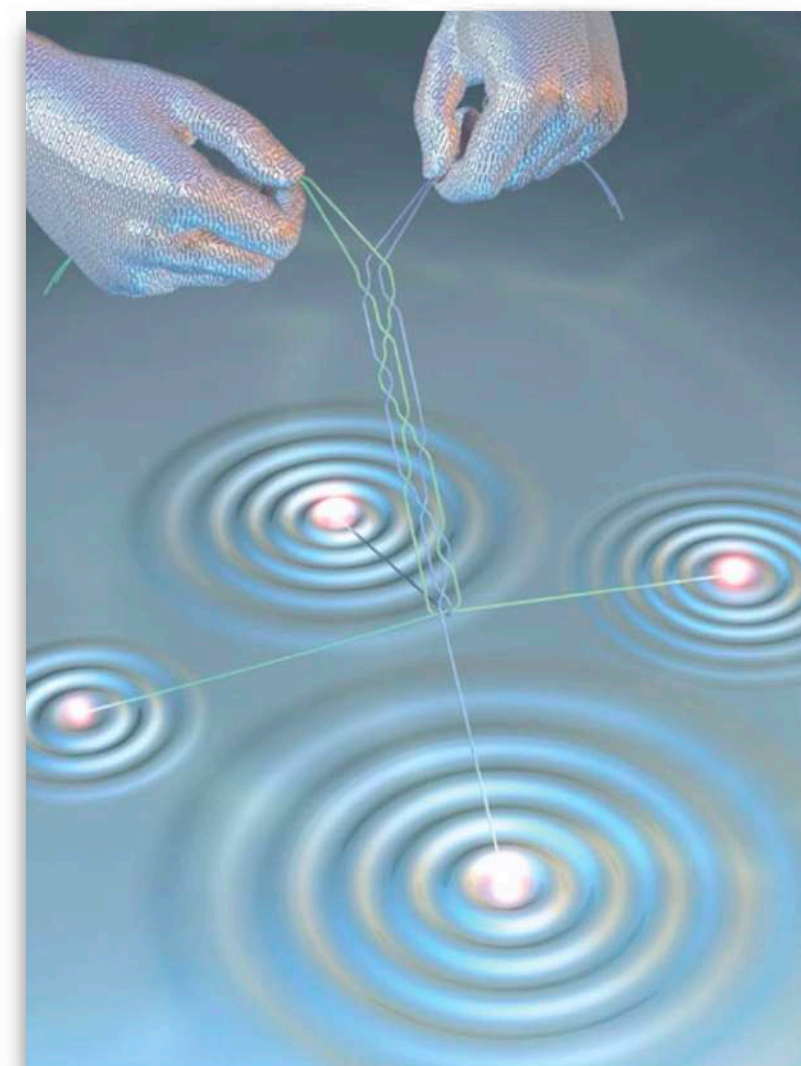
$$U_{23}|1, 1\rangle = \frac{1}{\sqrt{2}} [|1, 1\rangle - i|0, 0\rangle].$$

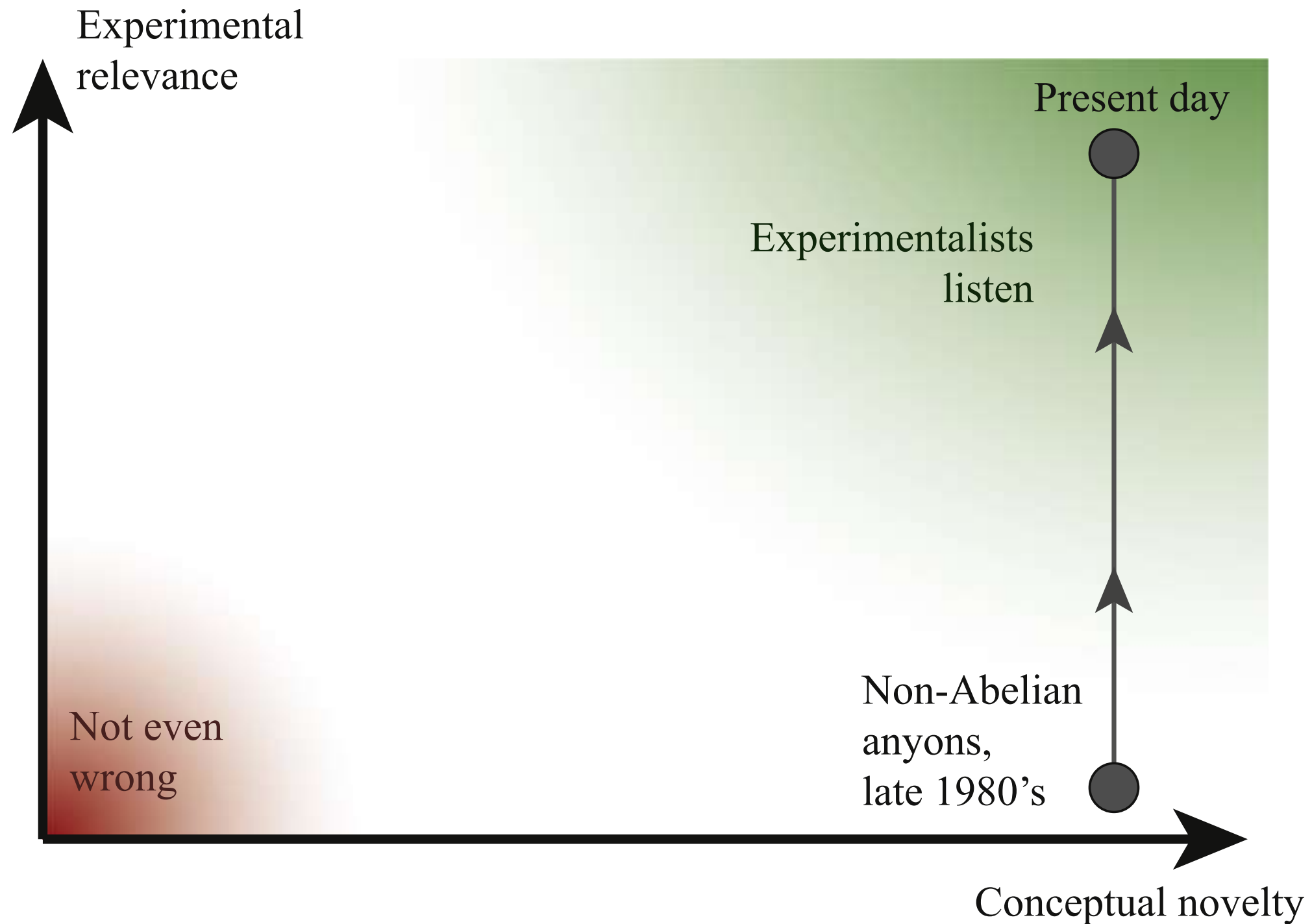
- Majorana exchange produces a superposition state of **different number states**.
- Whenever two exchanges involve the same Majorana zero mode, the braid operators do not commute.
- The final state of the system after exchanging several pairs of Majorana zero modes depends on the order of the exchange operations, much like braiding cords in a necklace, hence the name non-Abelian "braiding"

$$[U_{n-1,n} U_{n,n+1}] = i\gamma_{n-1}\gamma_{n+1}.$$



- *Non-Abelian anyons and topological quantum computation*, Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma, Rev. Mod. Phys. 80, 1083, 2008
- *Non-Abelian states of matter*, Ady Stern, Nature 464, 187, 2010
- *Majorana zero modes and topological quantum computation*, Sankar Das Sarma, Michael Freedman and Chetan Nayak, Npj Quantum Information, (2015) 15001.





Jason Alicea and Ady Stern, “Designer non-Abelian anyon platforms: from Majorana to Fibonacci”,
Physica. Scripta. **2015**, 014006, 2015

Challenging experimentally:

- electrons are spin-degenerate so we must freeze out half of the degrees of freedom to have an effective spinless system.
- p-wave superconductors are very rare in nature.

Clever proposals that overcome these challenges have the same three main ingredients:

1. Instead of using intrinsic superconductivity use the superconducting proximity effect.
2. Time-reversal symmetry breaking
3. Spin-orbit coupling.

Allows to use
standard
materials!!!



SINGLET PHASE (EVEN FERMION PARITY)

$$U \ll \Delta \quad \text{BCS-like singlet} \\ |S\rangle = u|0\rangle - v|\uparrow\downarrow\rangle$$

$$U \gtrsim \Delta \quad \text{Kondo-like YSR} \\ \text{singlets}$$

PARITY-DEGENERACY BOUNDARY: ZERO ENERGY CROSSINGS

$$T_K \sim \Delta$$

DOUBLET PHASE (ODD FERMION PARITY)

$$|D\rangle = |\uparrow\rangle, |\downarrow\rangle$$

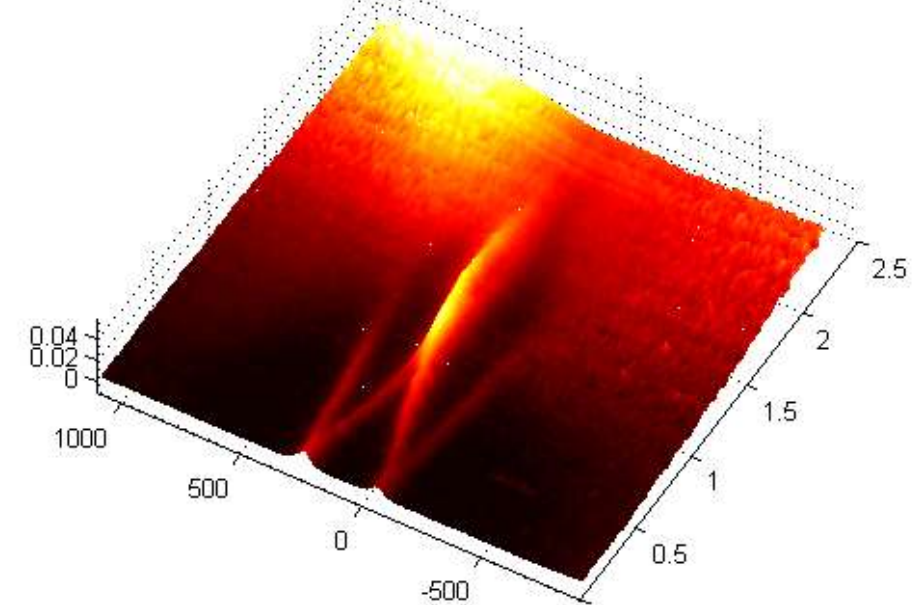
nature
nanotechnology

ARTICLES

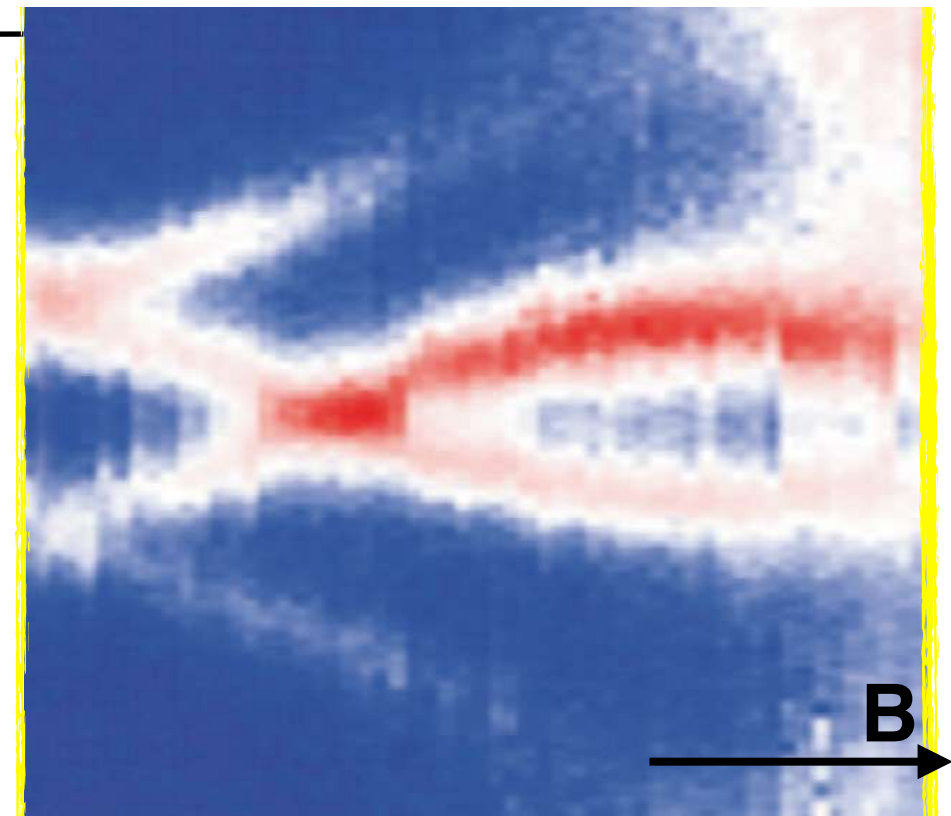
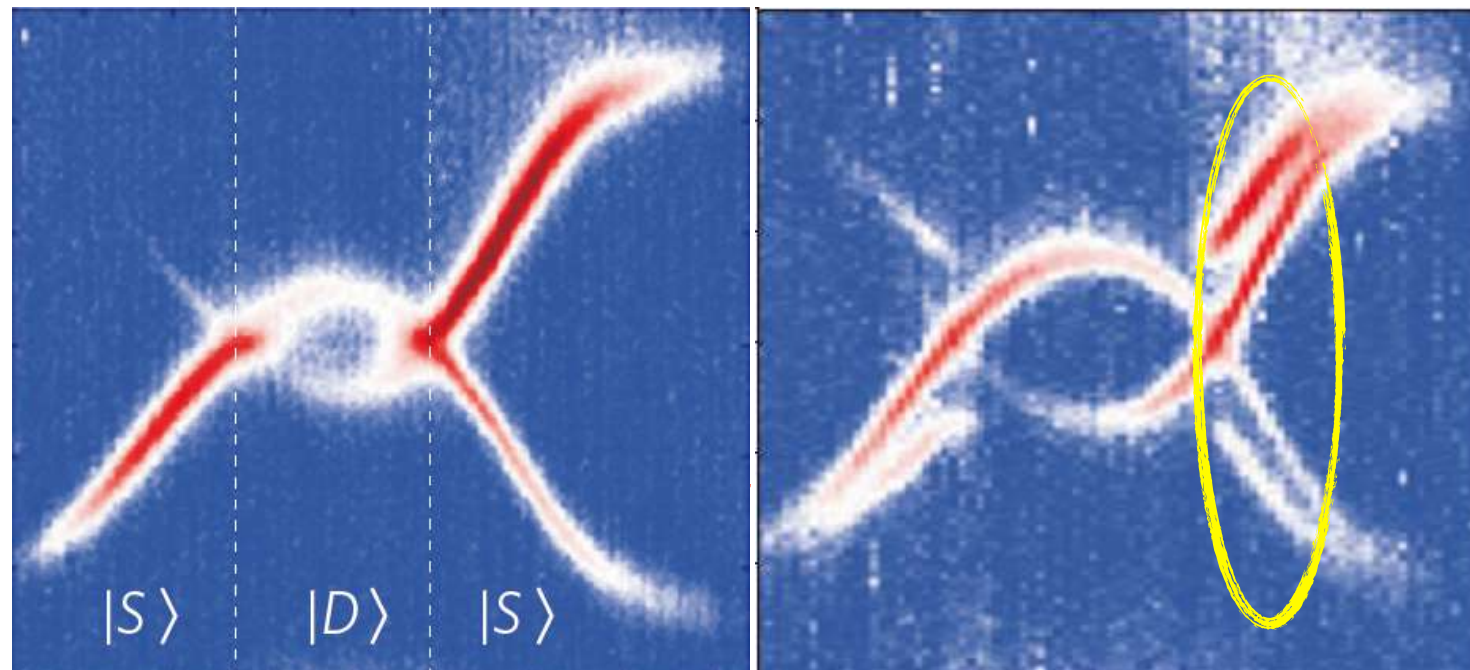
PUBLISHED ONLINE: 15 DECEMBER 2013 | DOI: 10.1038/NNANO.2013.267

Spin-resolved Andreev levels and parity crossings in hybrid superconductor-semiconductor nanostructures

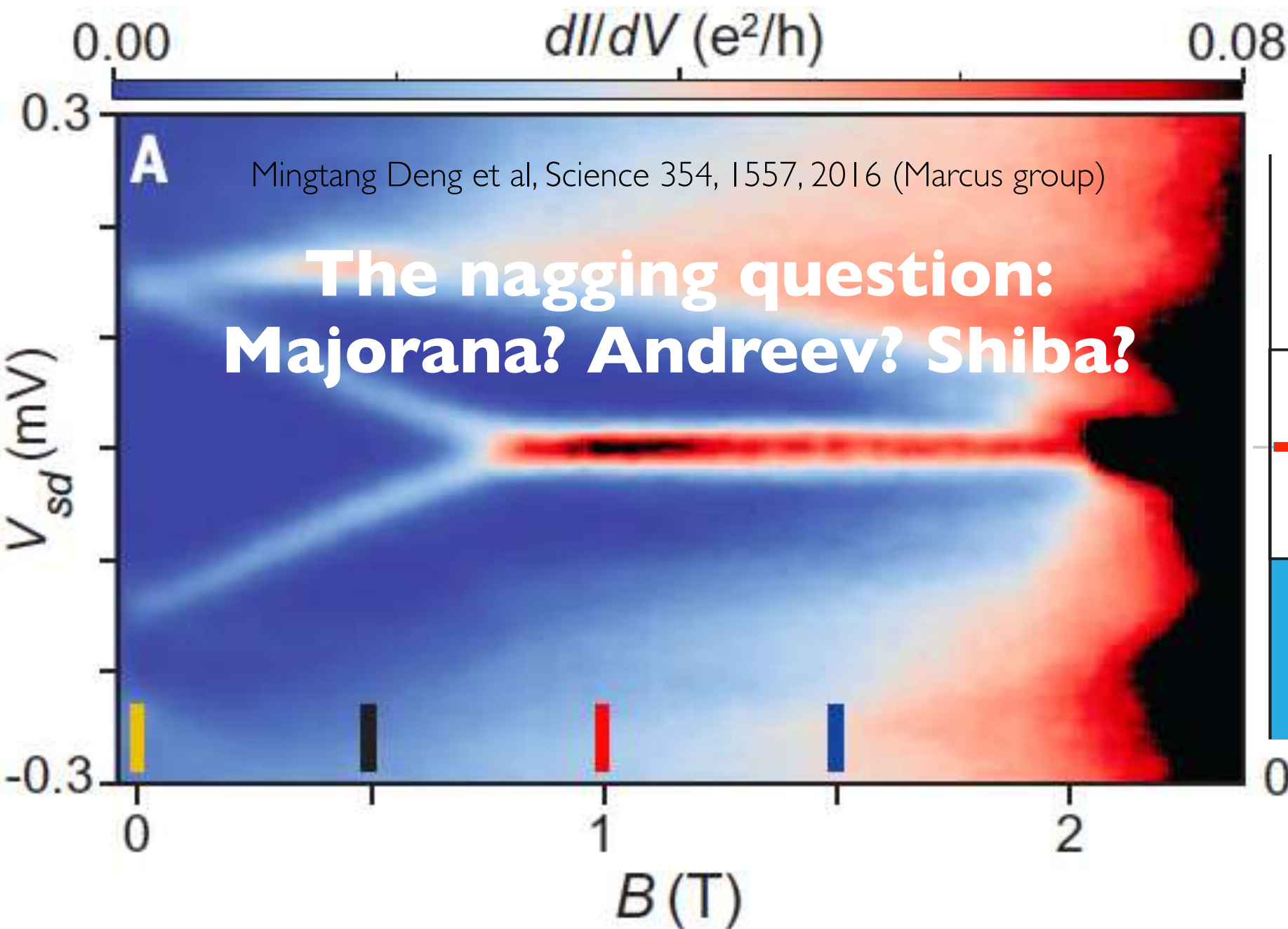
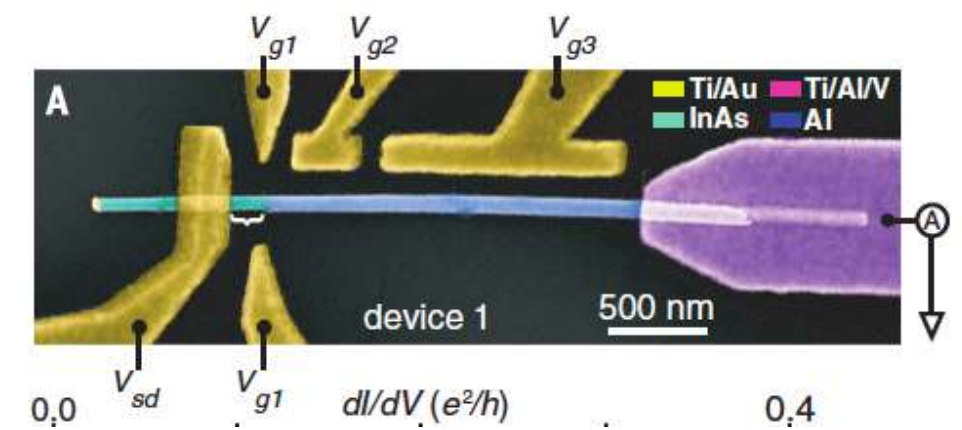
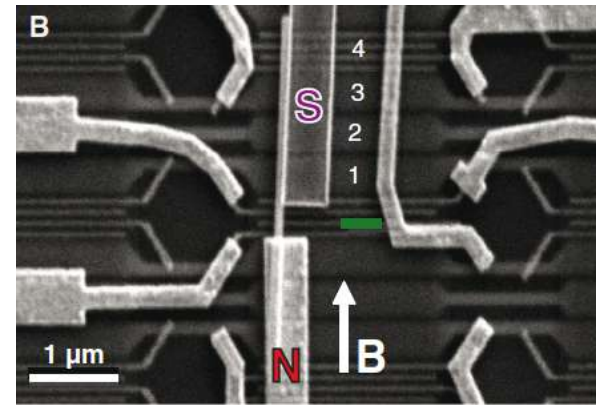
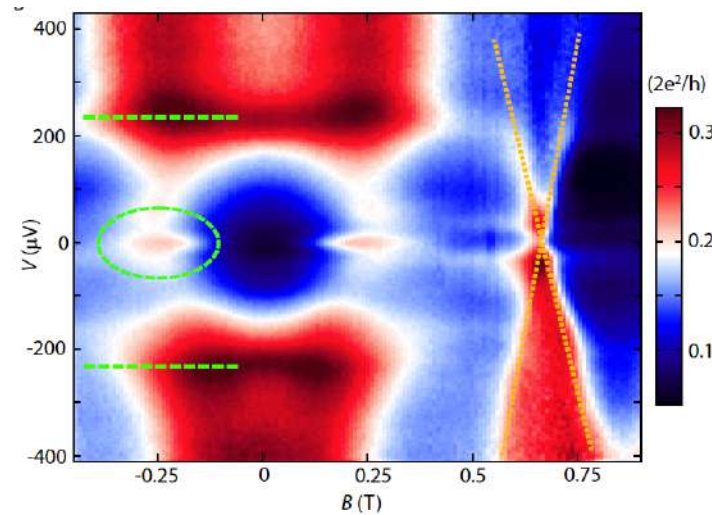
Eduardo J. H. Lee¹, Xiaocheng Jiang², Manuel Houzet¹, Ramón Aguado³, Charles M. Lieber² and Silvano De Franceschi^{1*}



ZEEMAN-INDUCED ZERO ENERGY CROSSINGS

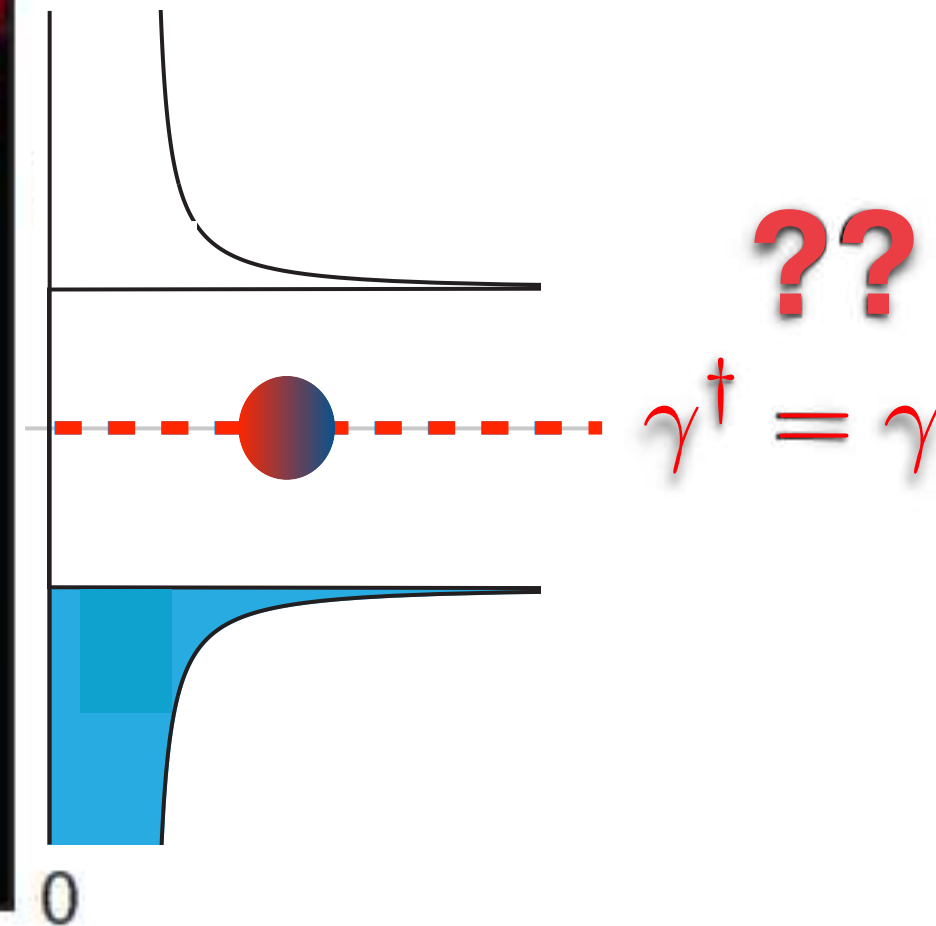


Mourik et al, Science **336**, 1003, 2012 (Kouwenhoven, Delft)



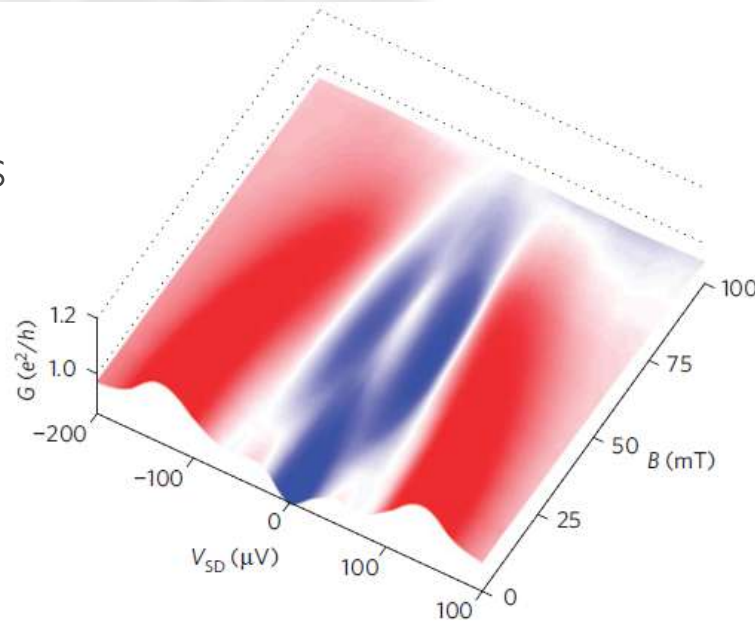
Mingtang Deng et al, Science 354, 1557, 2016 (Marcus group)

**The nagging question:
Majorana? Andreev? Shiba?**



MAJORANA INTERPRETATION

Das et al, Nature Physics
8, 887 (2012)

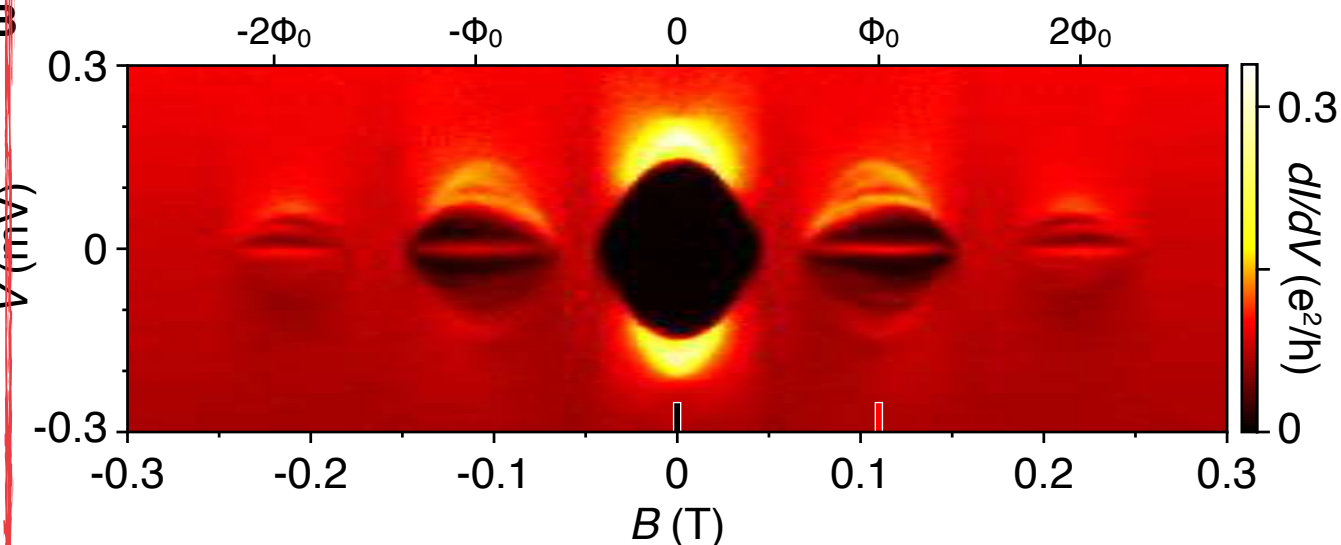
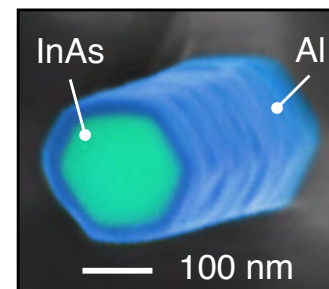


RESEARCH ARTICLE

TOPOLOGICAL MATTER

Flux-induced topological superconductivity in full-shell nanowires

S. Vaitiekėnas¹, G. W. Winkler², B. van Heck², T. Karzig², M.-T. Deng¹, K. Flensberg¹, L. I. Glazman³, C. Nayak², P. Krogstrup¹, R. M. Lutchyn^{2*}, C. M. Marcus^{1*}



QUANTUM DOT INTERPRETATION

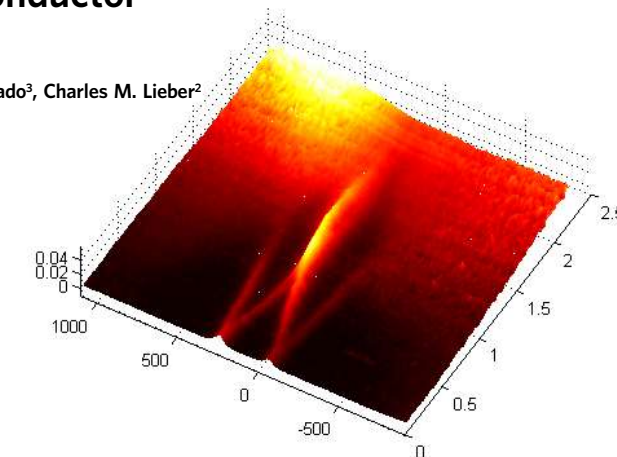
nature
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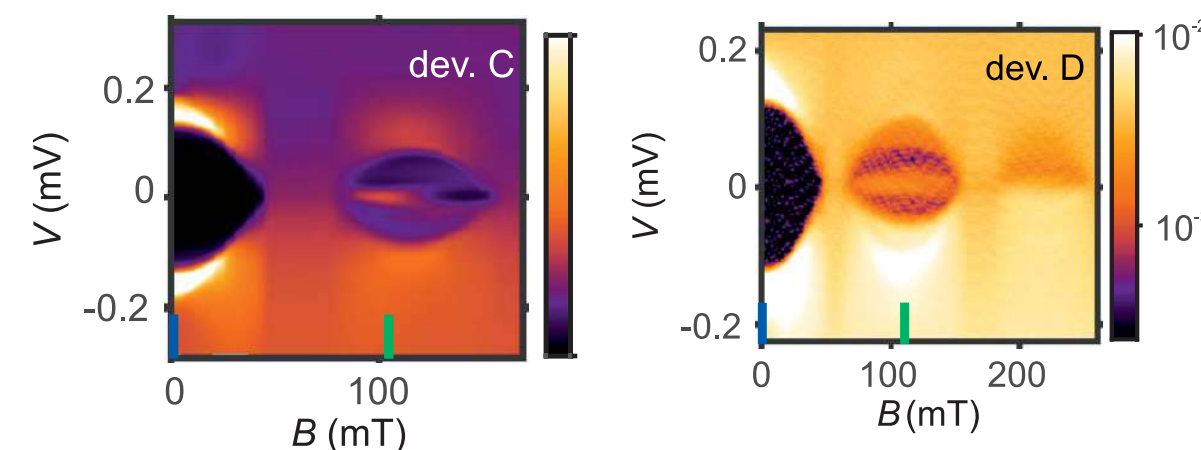
Eduardo J. H. Lee¹, Xiaocheng Jiang², Manuel Houzet¹, Ramón Aguado³, Charles M. Lieber² and Silvano De Franceschi^{1*}



MESOSCOPIC PHYSICS

Nontopological zero-bias peaks in full-shell nanowires induced by flux-tunable Andreev states

Marco Valentini^{1*}, Fernando Peñaranda², Andrea Hofmann^{1†}, Matthias Brauns^{1†}, Robert Hauschild¹, Peter Krogstrup³, Pablo San-Jose², Elsa Prada^{2,4}, Ramón Aguado^{2*}, Georgios Katsaros^{1*}



From Andreev to Majorana bound states in hybrid superconductor–semiconductor nanowires

Elsa Prada¹✉, Pablo San-Jose², Michiel W. A. de Moor³, Attila Geresdi³, Eduardo J. H. Lee¹, Jelena Klinovaja⁴, Daniel Loss⁴, Jesper Nygård⁵, Ramón Aguado² and Leo P. Kouwenhoven^{3,6}

Type	Subtype	Bulk topology	Spatial Overlap of Majorana components	Spatial extension of Majorana components	Zero energy pinning	Non-Abelian braiding
ABSs	Standard: $SO=0, V_Z=0$	trivial	complete	spread across junction / normal region	no	no
	$SO\neq 0, V_Z < V_Z^c$	trivial	partial	spread across junction / normal region	no	no
	strong $SO, V_Z < V_Z^c$	trivial	partial	spread across junction / normal region	only vs. V_Z , rest fine-tuned	no
	coupled multiband + short-range inhomogeneity	trivial	high	spread across inhomogeneity	approximate	no
	Shiba state	trivial	complete	localized to impurity	no	no
MBSs	Long ($L \gg \xi_M, V_Z > V_Z^c$)	nontrivial	exponentially suppressed	localized to edges	yes	yes
	Short ($L \lesssim \xi_M, V_Z > V_Z^c$)	nontrivial	partial	localized to edges but overlapping	no (Majorana oscillations)	no
Smooth zero modes	Smoothly confined S	trivial	partial	localized to smooth edge	yes	yes (parametric)
	Smooth S'S/NS junction (nontopological MBS/ps-MBS/quasi-MBS/EP-MBS)	trivial	partial	localized to smooth junction	yes	yes (parametric)

Nature | Vol 614 | 16 February 2023 | 445

Article

Realization of a minimal Kitaev chain in coupled quantum dots

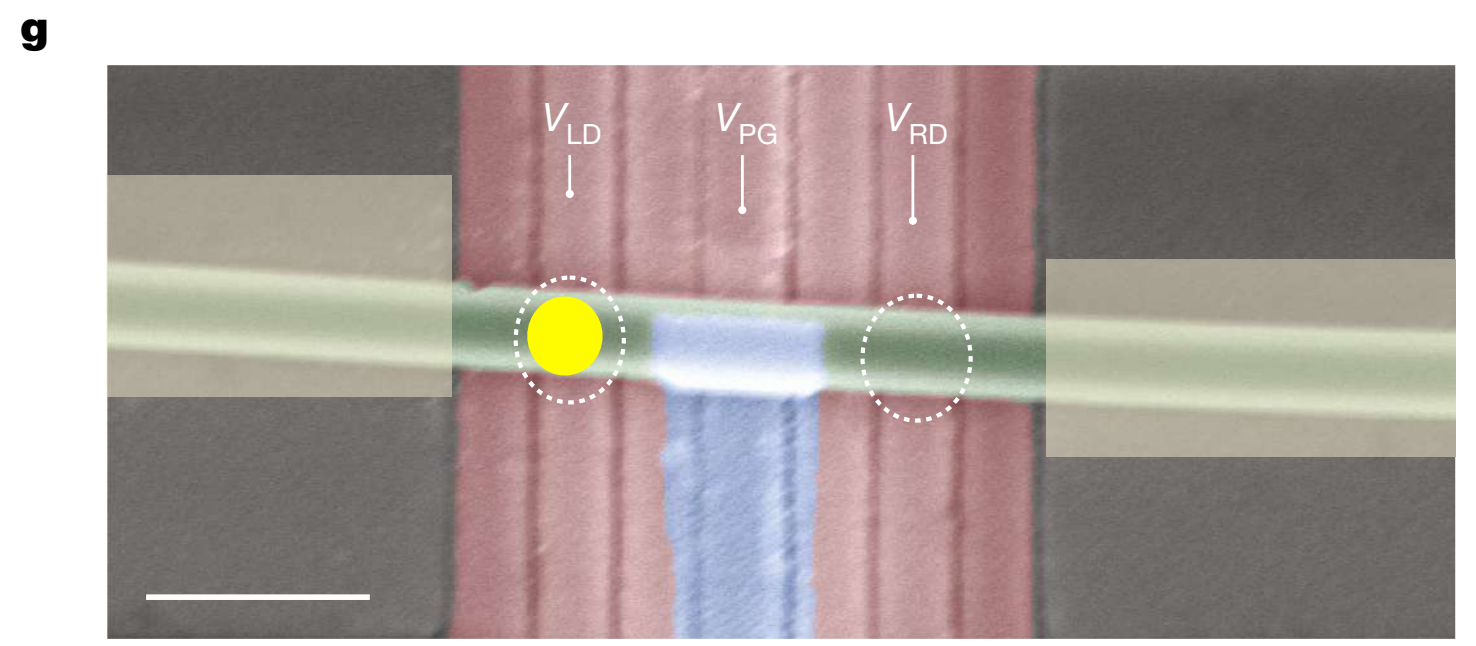
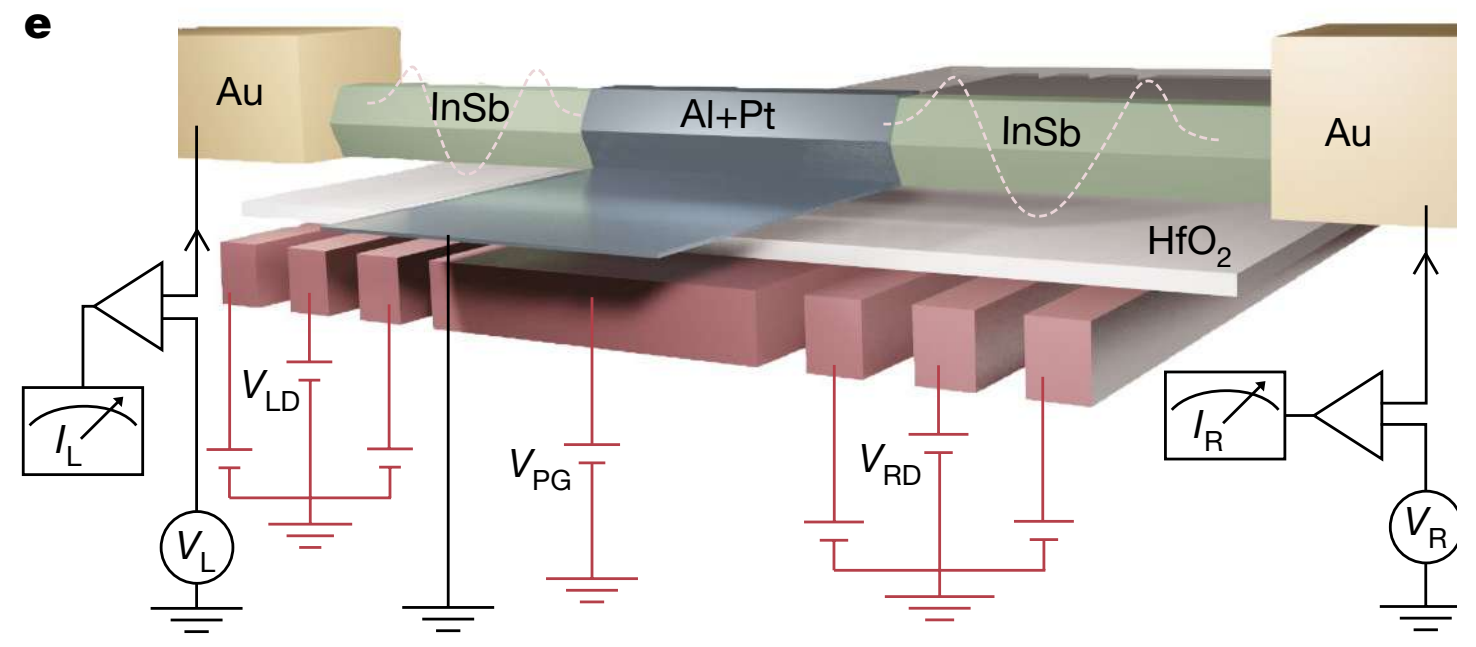
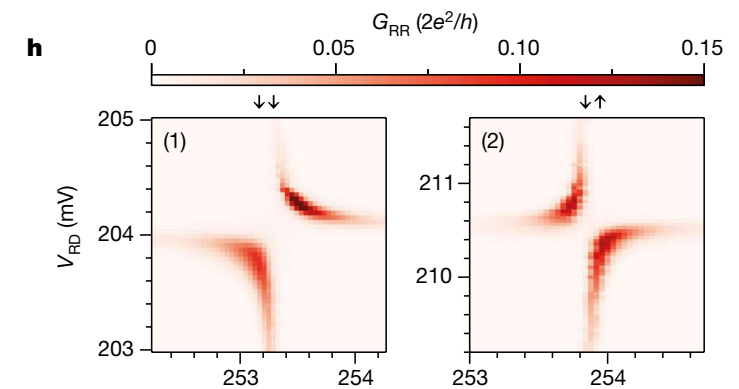
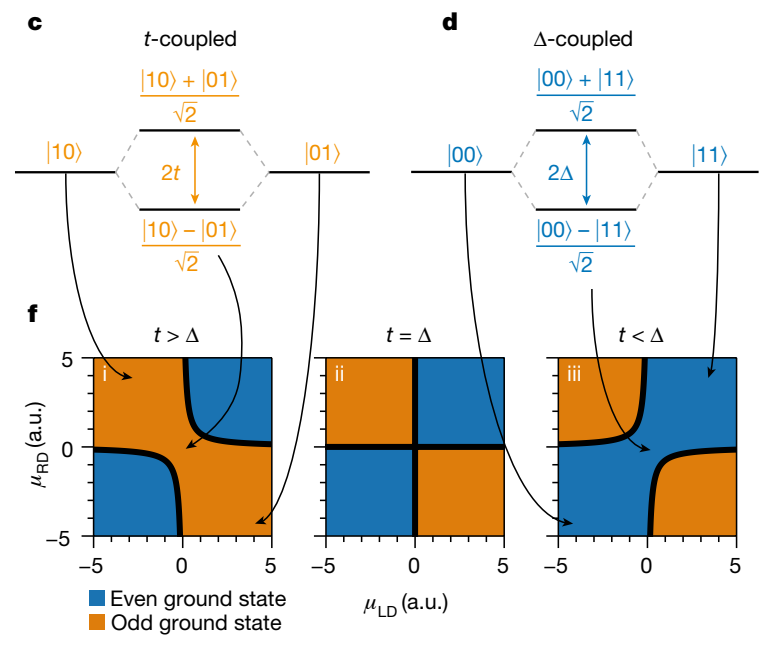
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$$H_K = \mu_1 c_1^\dagger c_1 + \mu_2 c_2^\dagger c_2 + (t c_1^\dagger c_2 + \Delta c_1 c_2 + \text{H.c.})$$



LEFT QUANTUM

RIGHT QUANTUM DOT



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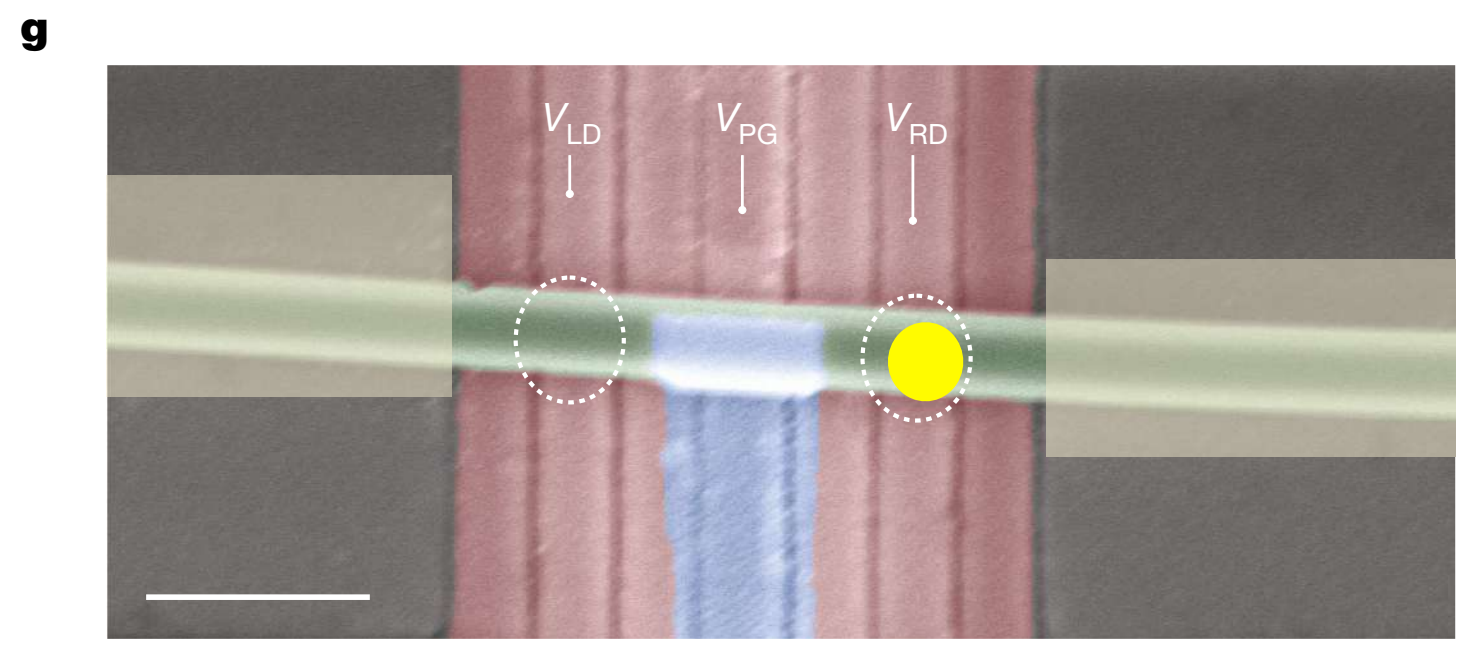
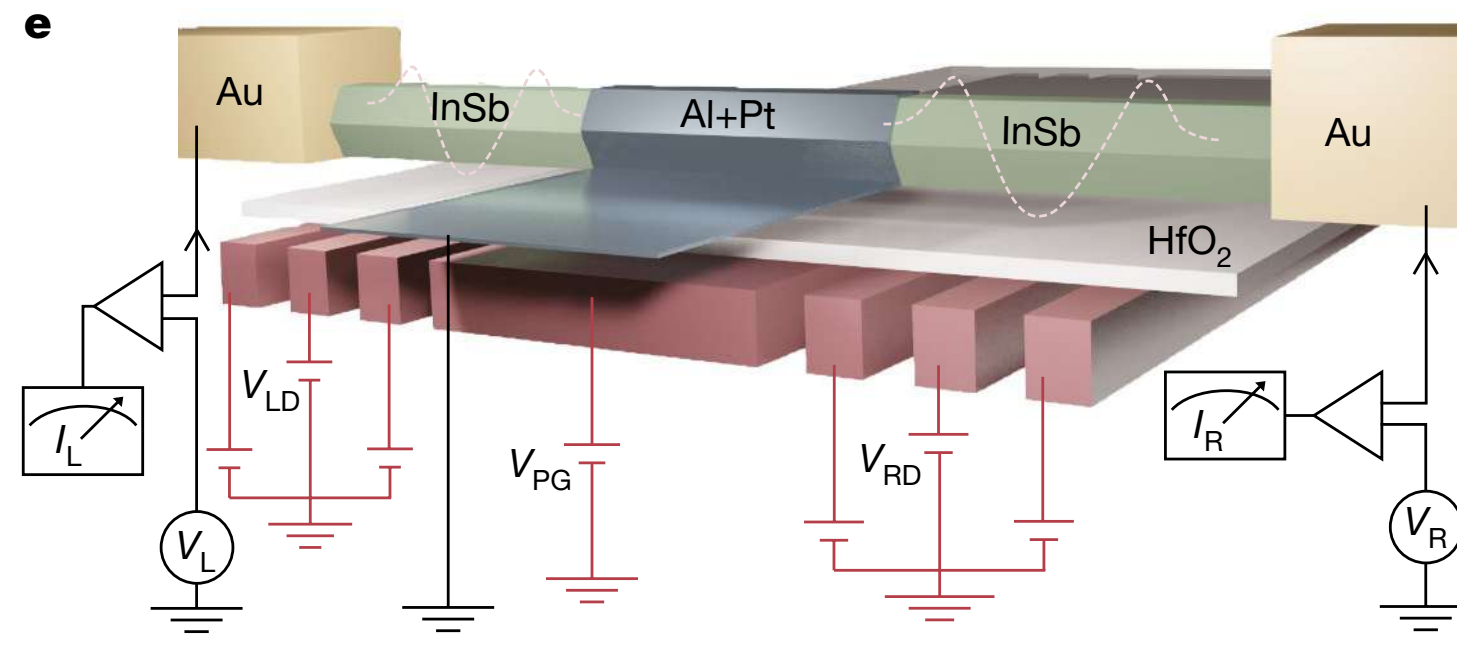
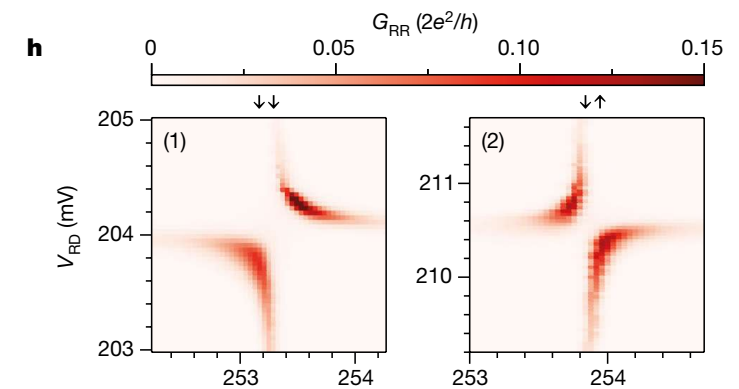
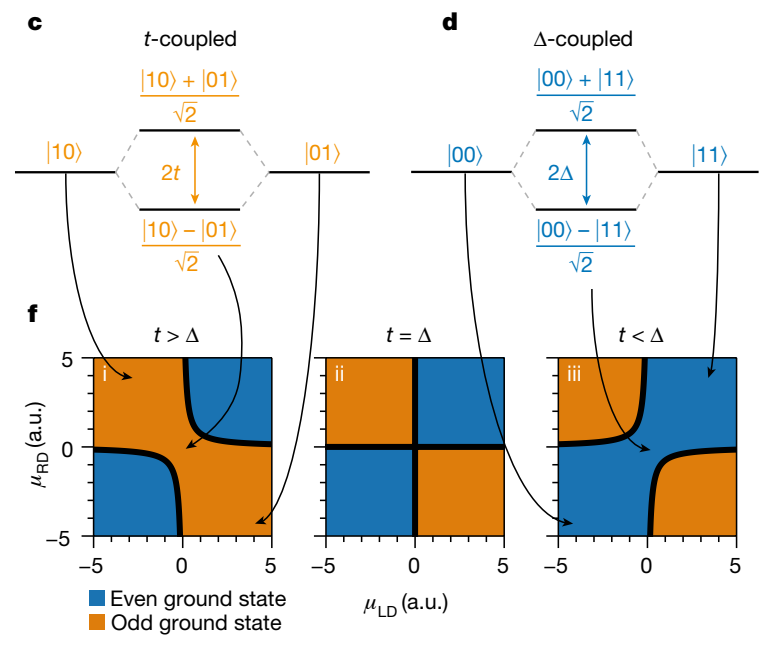
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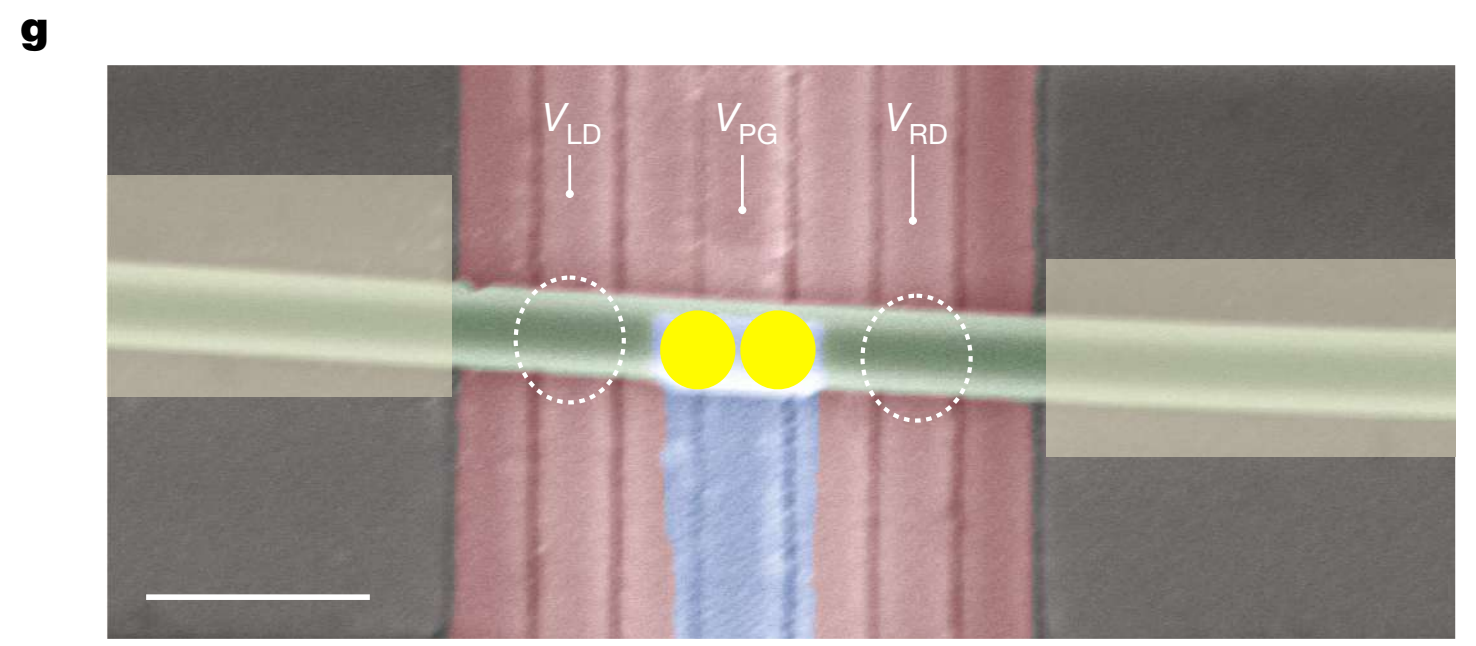
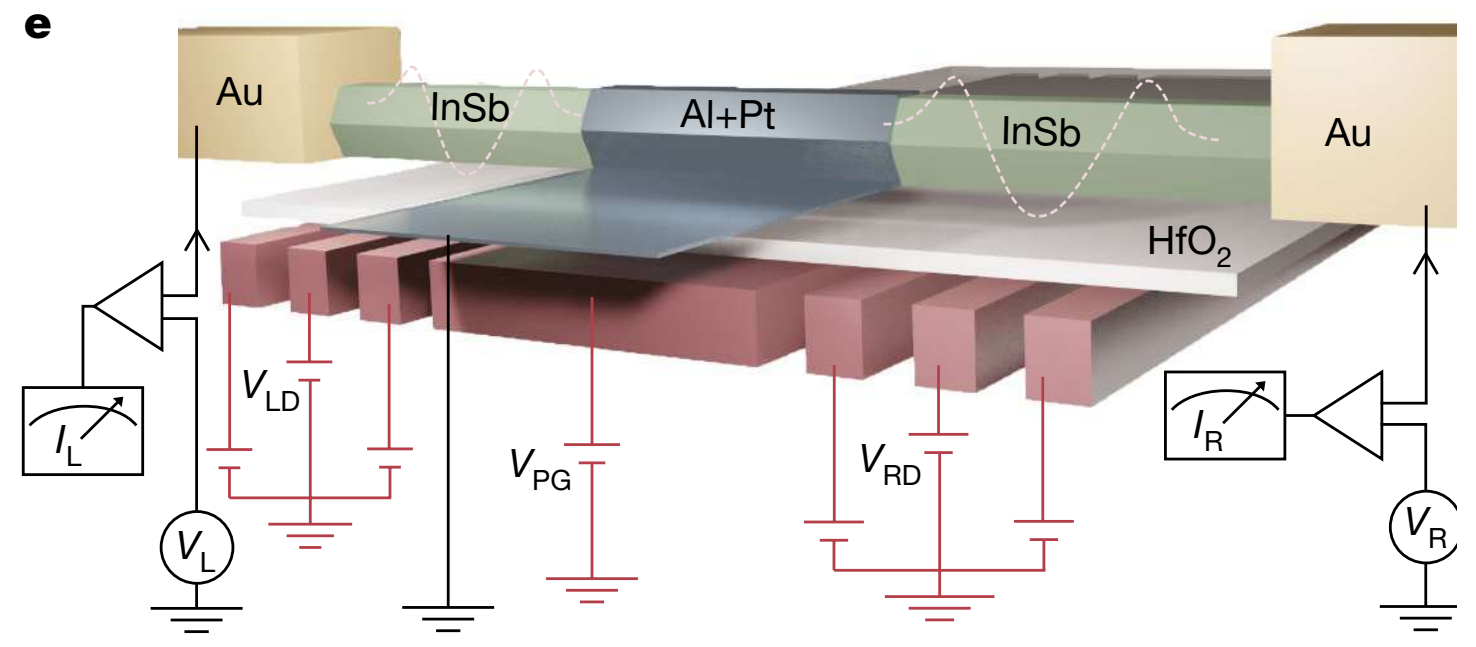
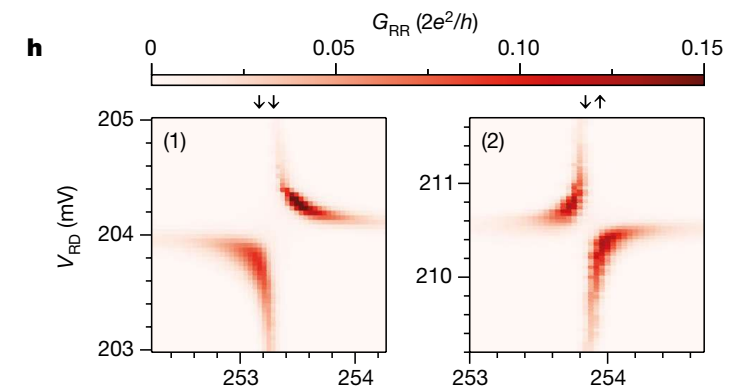
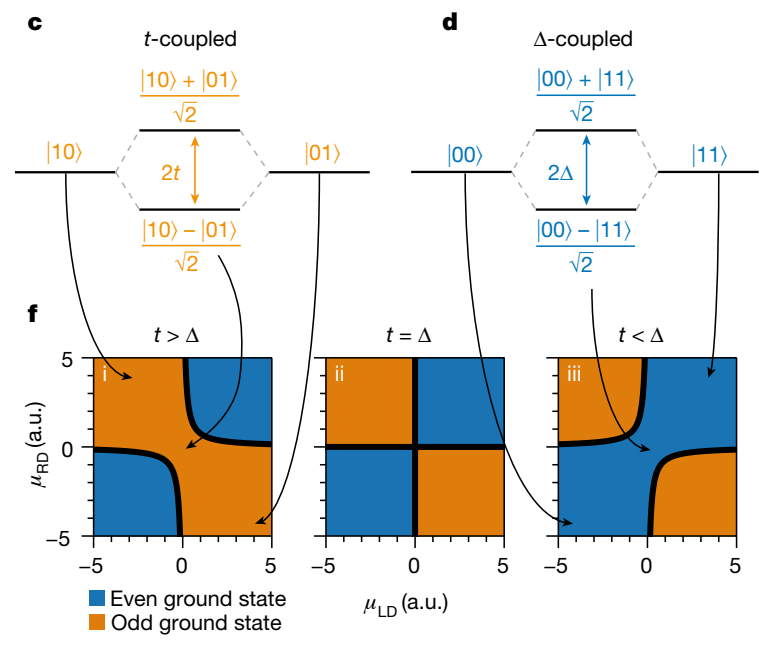
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RIGHT QUANTUM DOT



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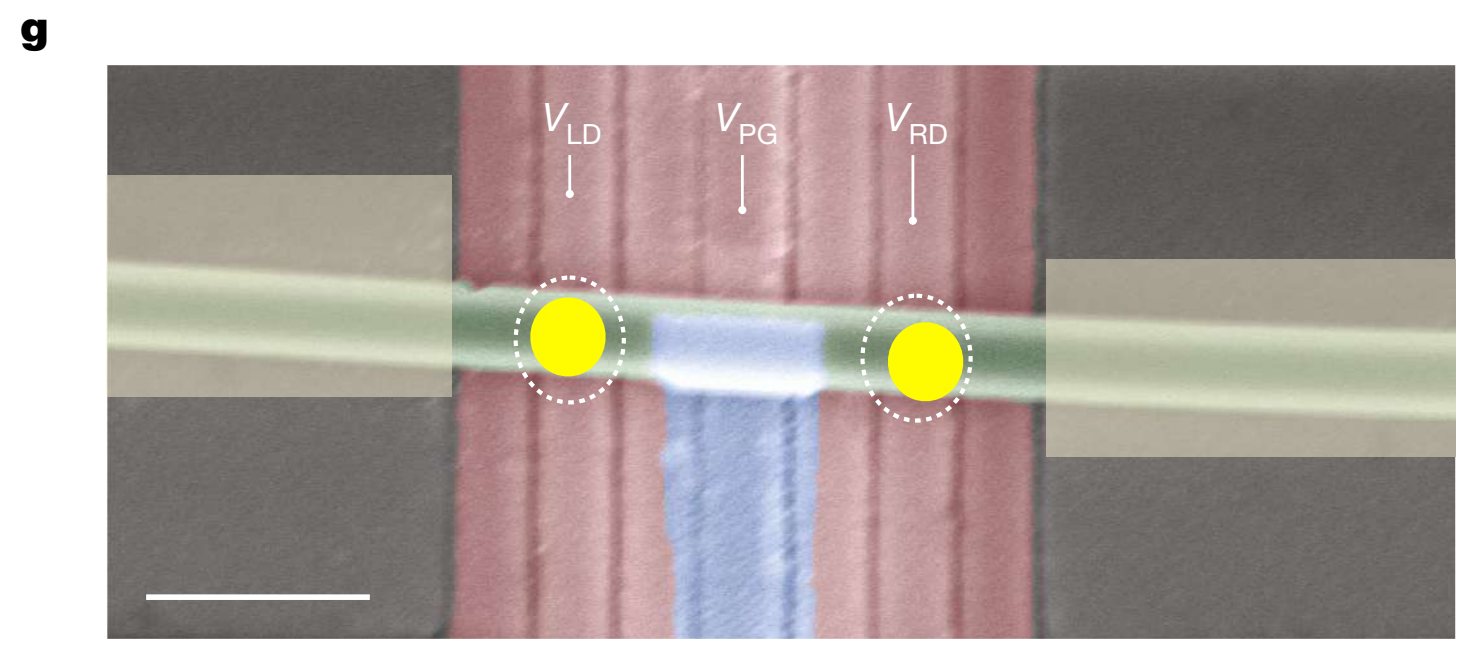
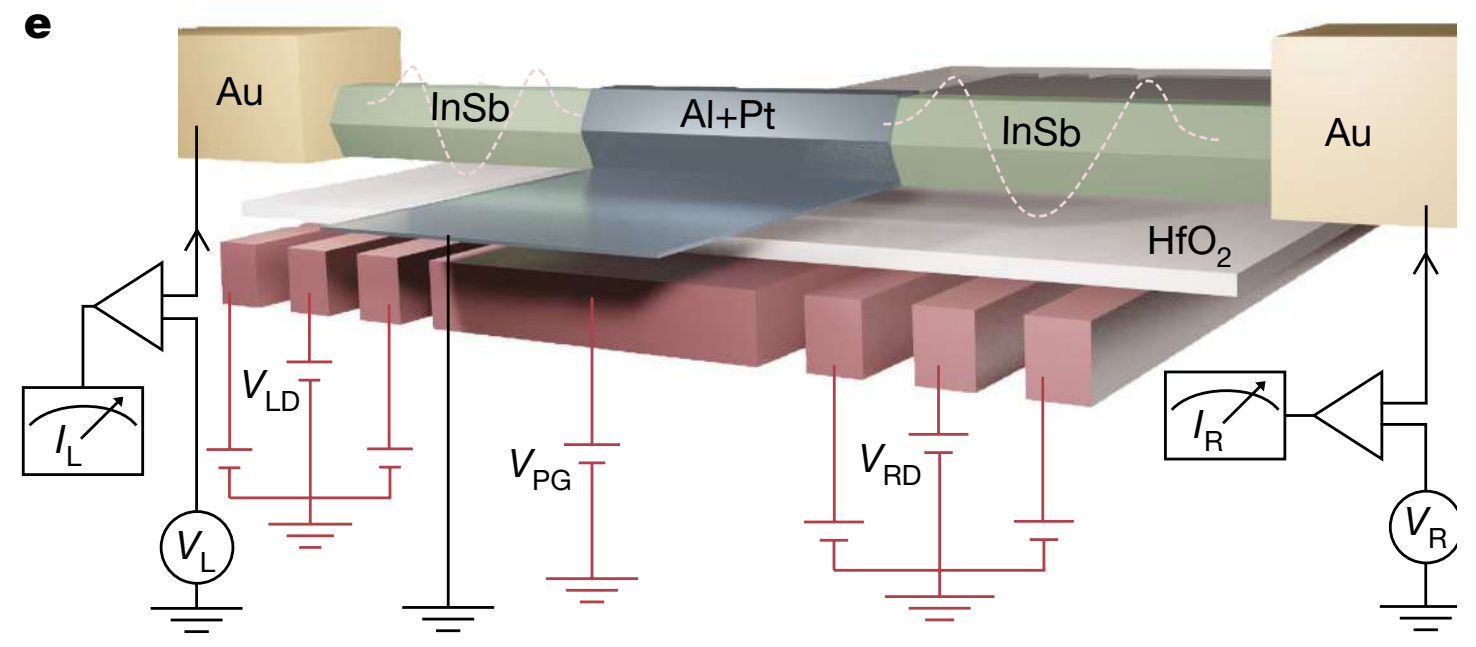
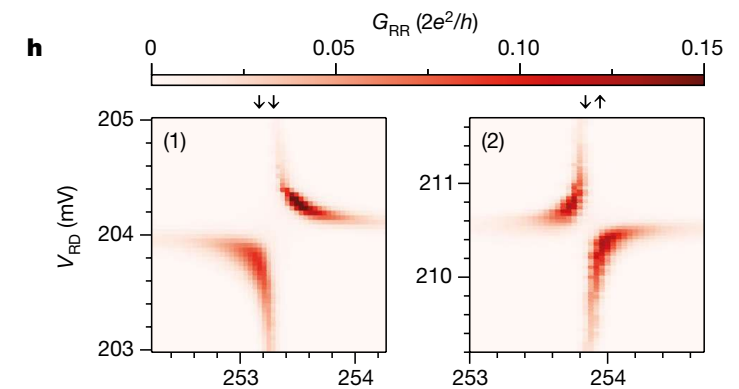
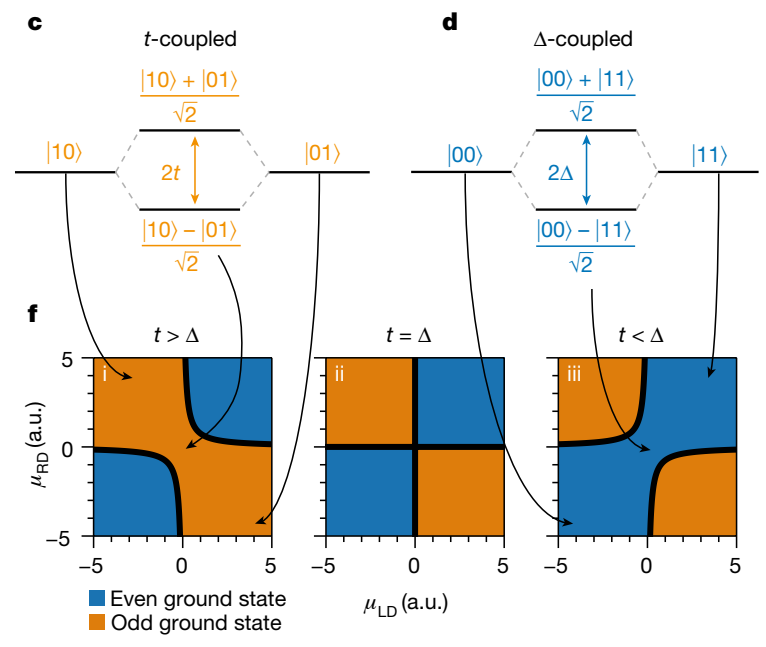
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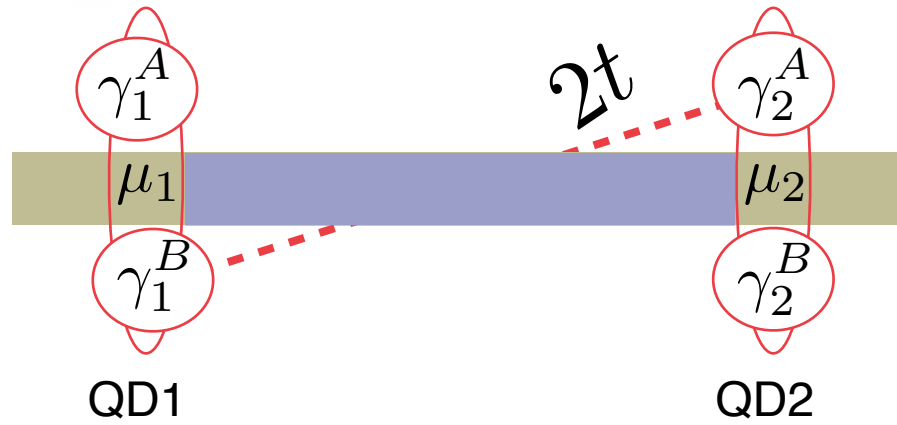
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LEFT QUANTUM

RIGHT QUANTUM DOT



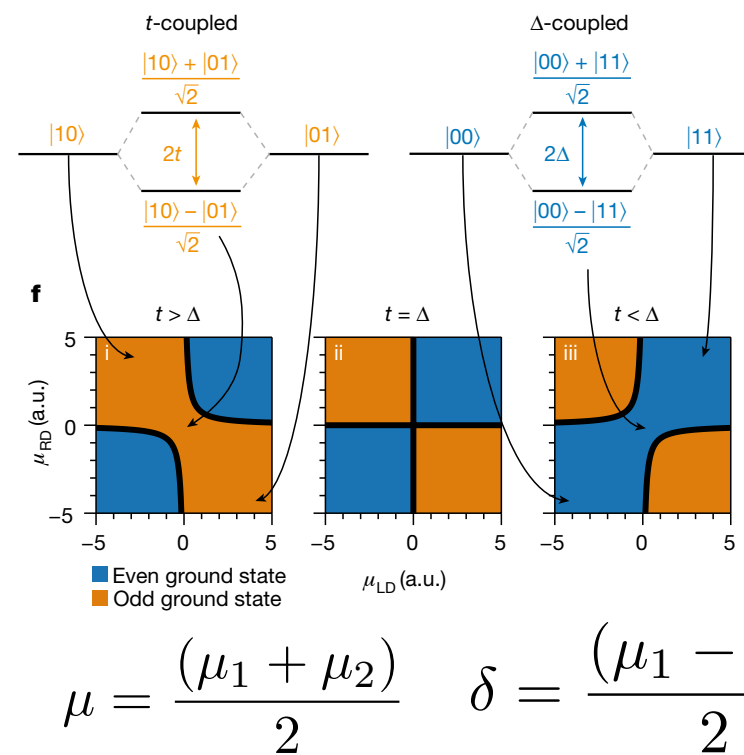


$$H_K = \mu_1 c_1^\dagger c_1 + \mu_2 c_2^\dagger c_2 + (t c_1^\dagger c_2 + \Delta c_1 c_2 + \text{H.c.})$$

Sweet spot

$$\mu_1 = \mu_2 = 0 \quad \Delta = t$$

$$\epsilon_O^\pm = \epsilon_E^\pm = \pm t = \pm \Delta$$



PHYSICAL REVIEW B **86**, 134528 (2012)

Parity qubits and poor man's Majorana bound states in double quantum dots

Martin Leijnse and Karsten Flensberg

Degenerate even and odd ground states with zero energy excitations

$$\epsilon_O^- = \epsilon_E^-$$

$$\mu_1 \mu_2 = \Delta^2 - t^2$$

Odd

$$\epsilon_O^\pm = -\mu \pm \sqrt{t^2 + \delta^2}$$

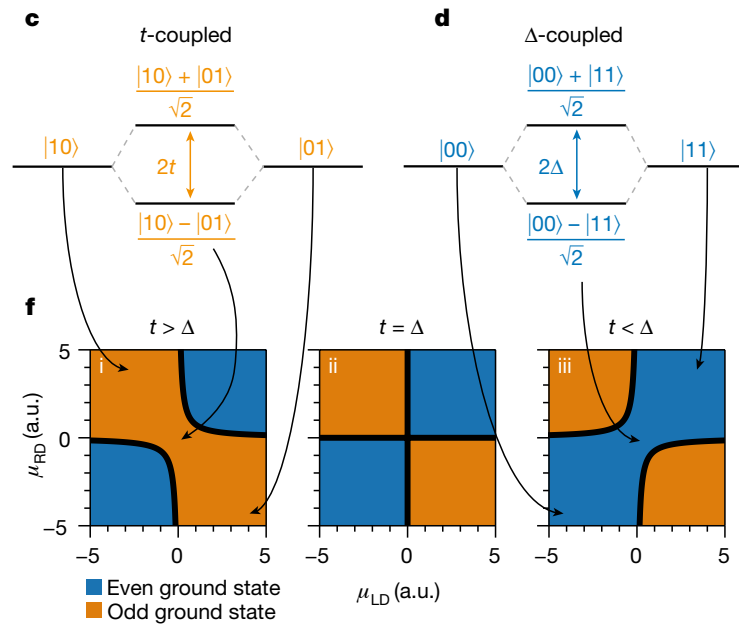
$$|O\pm\rangle = \frac{1}{2}(|10\rangle \pm |01\rangle)$$

Even

$$\epsilon_E^\pm = -\mu \pm \sqrt{\Delta^2 + \mu^2}$$

$$|E\pm\rangle = \frac{1}{2}(|00\rangle \pm |11\rangle)$$

$$H_K = \mu_1 c_1^\dagger c_1 + \mu_2 c_2^\dagger c_2 + (t c_1^\dagger c_2 + \Delta c_1 c_2 + \text{H.c.})$$



Many body basis

$$|10\rangle, |01\rangle, |00\rangle, |11\rangle$$

$$H_K = \begin{pmatrix} \mu_1 & t & 0 & 0 \\ t & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & \Delta \\ 0 & 0 & \Delta & \mu_1 + \mu_2 \end{pmatrix}$$

Doublet

$$H_D = \begin{pmatrix} \epsilon_\uparrow & 0 & 0 & 0 \\ 0 & \epsilon_\downarrow & 0 & 0 \\ 0 & 0 & 0 & \Gamma_S \\ 0 & 0 & \Gamma_S & \epsilon_\uparrow + \epsilon_\downarrow + U \end{pmatrix}$$

Singlet

Analogy between a local spin qubit and a nonlocal charge qubit based on DQDs in the language of the large gap superconducting Anderson model

